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Resource: Digital Signal Processing
Prof. Alan V. Oppenheim

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INTRODUCTION

1. Lesson 1 - 17 minutes

This lecture serves as an introduction to the course and is intended to provide an indication of the importance and scope of the field of digital signal processing. It is suggested that in addition to viewing the lecture you read the introduction to the text (pages 1-7) and casually peruse the text and manual to obtain a general picture of the scope of this course.

As mentioned in the preface this course assumes a previous exposure to linear system theory for continuous-time signals and systems including Fourier and Laplace Transforms. If it has been some time since you have used that material, you may want to spend a few hours reviewing this background. There is a long list of excellent texts on linear system theory and everyone has his favorite. I would suggest that you look through one that you are familiar with. Several that I have found useful are:

Cooper, G.R. and C.D. McGillem Methods of Signal and System Analysis Holt, Rinehart and Winston, Inc., New York 1967.

Lathi, B.P. Signals, Systems and Communication John Wiley and Sons, Inc., New York 1965.

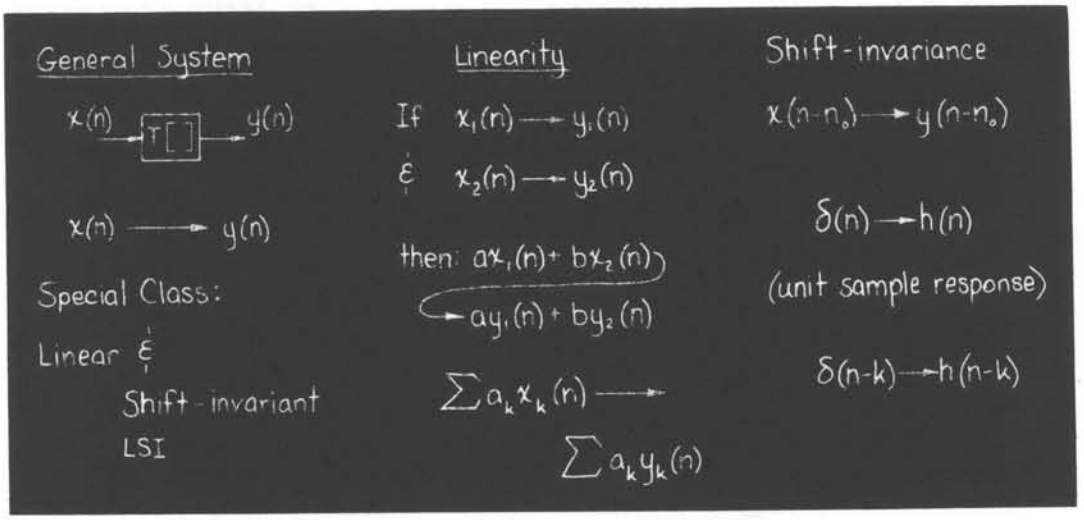
Oppenheim, A.V. and A.S. Willsky Signals and Systems Prentice-Hall, Inc., New Jersey 1983.

Papoulis, A The Fourier Integral and Its Applications McGraw-Hill Book Company, New York 1962.

DISCRETE-TIME SIGNALS AND SYSTEMS, PART 1

1. Lecture 2 - 36 minutes

a.



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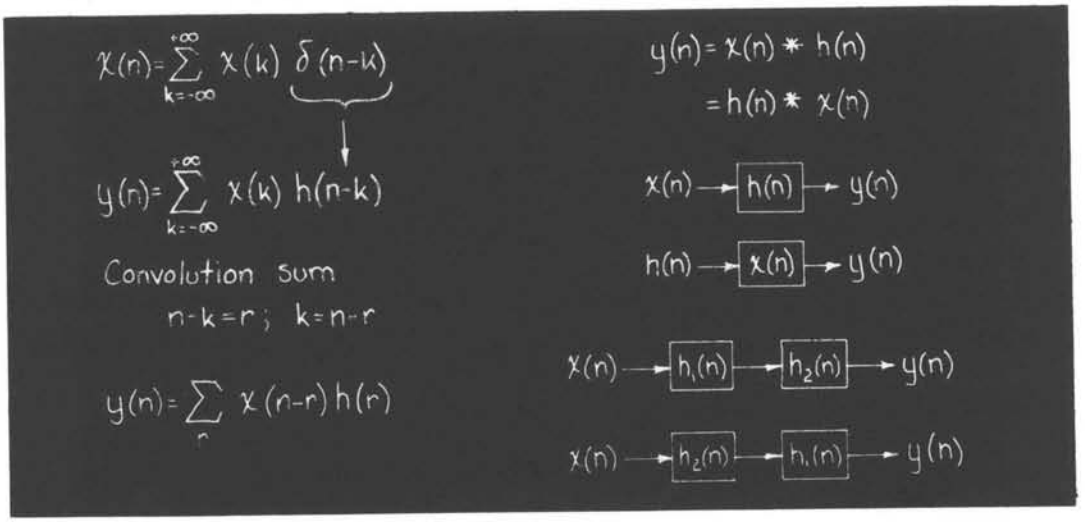

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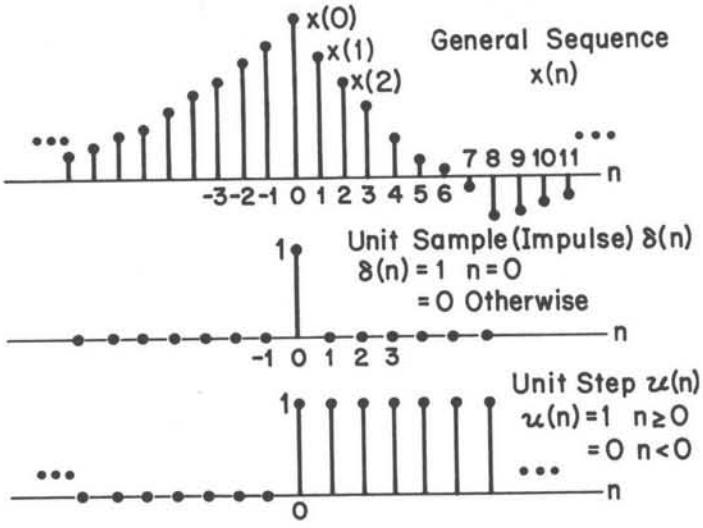
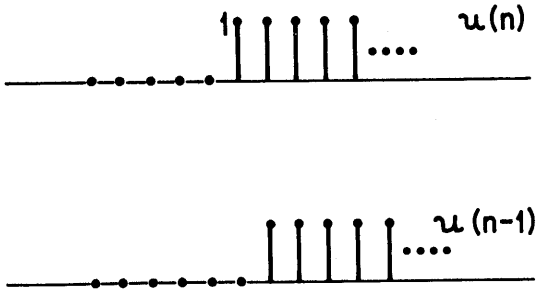


Graphical representation of Discrete-Time Signals

$$\delta(n) = u(n) - u(n-1)$$

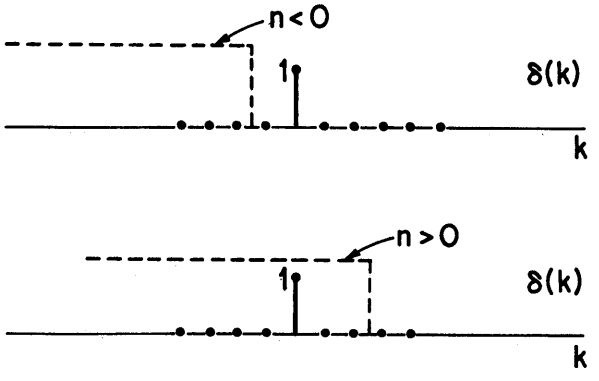
The unit-sample sequence in terms of the unit-step sequence.



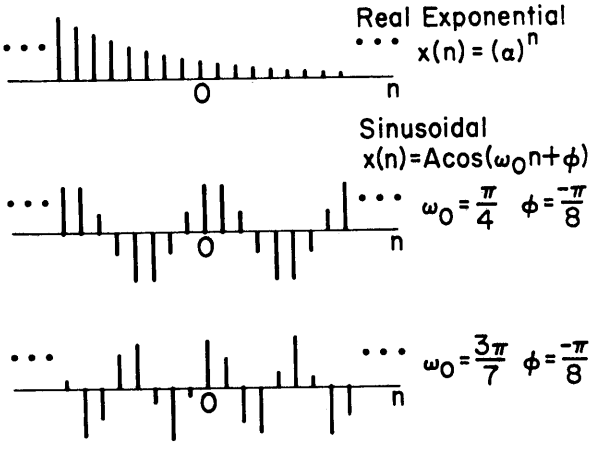
d.

$$u(n) = \sum_{k=-\infty}^{n} \delta(k)$$

The unit-step sequence in terms of the unit-sample sequence.

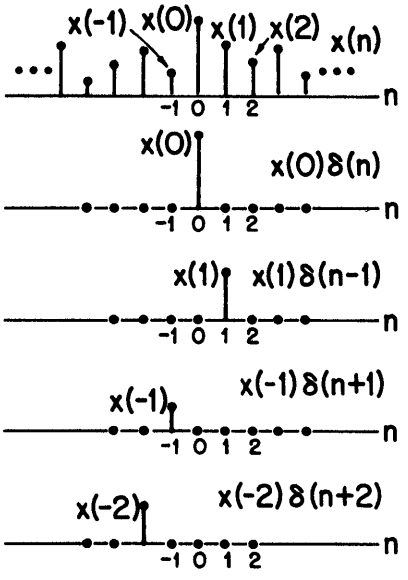


e.



Exponential and Sinusoidal sequences.

f.



$$x(n) = x(0)\delta(n)+x(1)\delta(n-1)+x(-1)\delta(n+1)+\cdots = \sum_{k=-\infty}^{+\infty} x(k)\delta(n-k)$$

Representation of an arbitrary sequence as a linear combination of delayed unit samples.

g.

$$y(n)= \sum_{k=-\infty}^{+\infty} x(k)\, h(n-k)$$

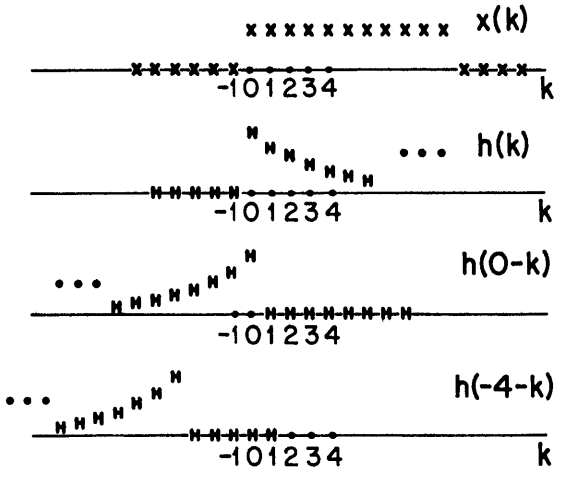


Illustration of folding and shifting for linear convolution.

h.

2. Correction

In the lecture I indicate that the sinusoidal sequence $A\cos(\omega_0 n + \phi)$ with $\omega_0 = 3\pi/7$ and $\phi = -\pi/8$ is not periodic. In fact it is periodic although not with a period of $2\pi/\omega_0$. (See problem 2.1(a)). For $\omega_0 = 3/7$ the sinusoidal sequence will not be periodic.

3. Comments

In this lecture we introduce the class of discrete-time signals and systems. The unit sample, unit step, exponential and sinusoidal sequences are basic sequences which play an important role in the analysis and representation of more complex sequences. The class of discrete-time systems that we focus on is the class of linear shift-invariant systems. The representation of this class of systems through the convolution sum and some properties of convolution are developed.

4. Reading

Text: Section 2.0 (page 8) through eq. (2.51) page 28 section 2.4.

5. Problems

Problem 2.1

Determine whether or not each of the following sequences is periodic. If your answer is yes, determine the period.

(a) $x(n) = A \cos(\frac{3\pi}{7} n - \frac{\pi}{8})$

(b) $x(n) = e^{j(n/8 - \pi)}$

Problem 2.2

A sequence x(n) is shown below. Express x(n) as a linear combination of weighted and delayed unit samples.

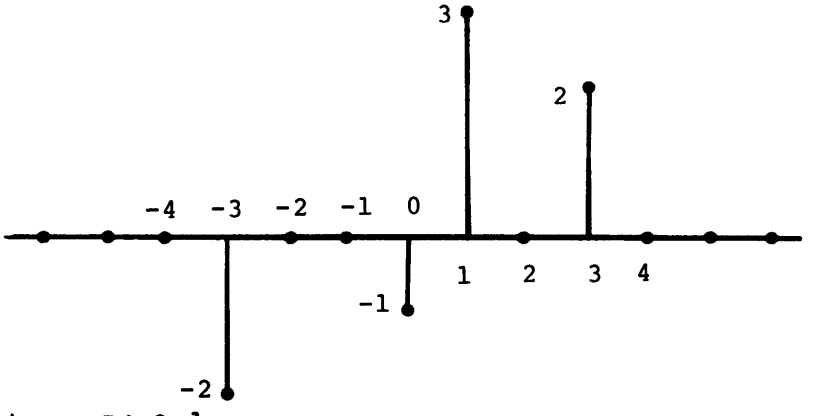


Figure P2.2-1

Problem 2.3

For each of the following systems, y(n) denotes the output and x(n) the input. Determine for each whether the specified input-output relationship is linear and/or shift-invariant.

(a) $y(n) = 2x(n) + 3$

(b) $y(n) = x(n) \sin(\frac{2\pi}{7} n + \frac{\pi}{6})$

(c) $y(n) = [x(n)]^2$

(d) $y(n) = \sum_{m=-\infty}^{n} x(m)$

Problem 2.4

For each of the following pairs of sequences, x(n) represents the input to an LSI system with unit-sample response h(n). Determine each output y(n). Sketch your results.

(a)

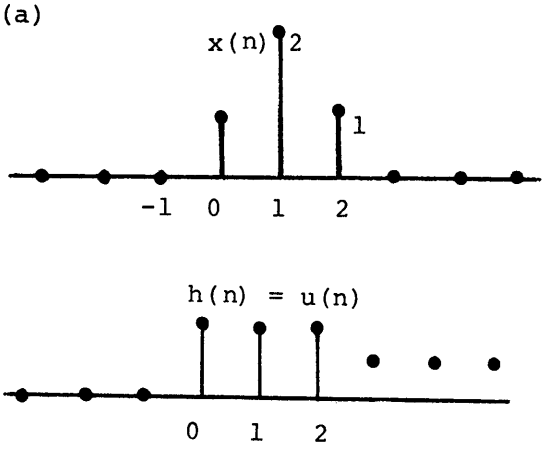


Figure P2.4-1

(b)

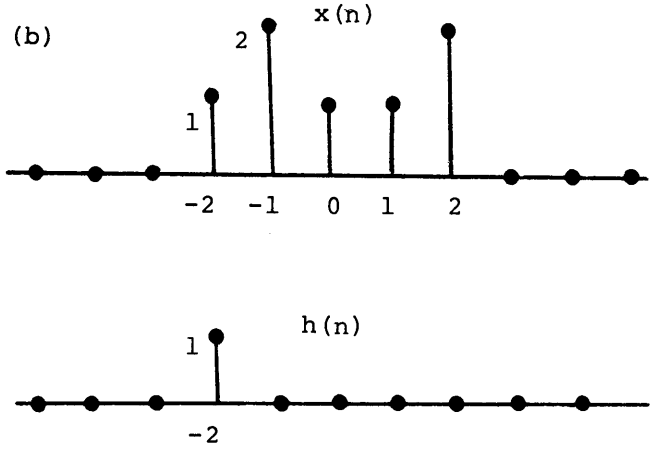
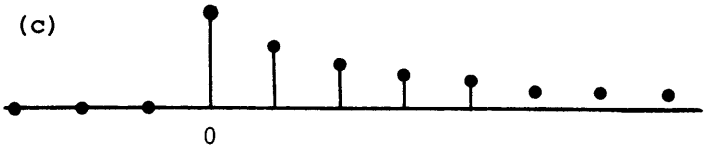


Figure P2.4-2

(c) $x(n) = \alpha^n u(n)$; $0 < \alpha < 1$



$h(n) = \beta^n u(n)$; $0 < \beta < 1$; $\beta \neq \alpha$

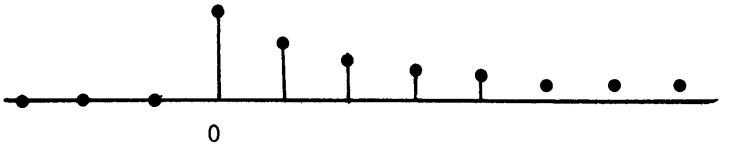
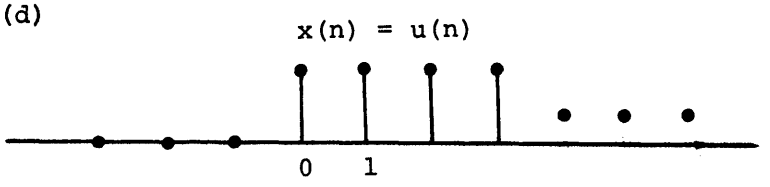


Figure P2.4-3

(d)



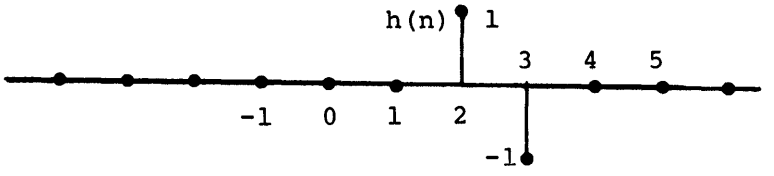


Figure P2.4-4

The following formulas may be useful:

$$\sum_{r=0}^{\infty} a^r = \frac{1}{1-a} \quad , \ |a| < 1$$

$$\sum_{r=0}^{N-1} a^r = \frac{1-a^N}{1-a} \quad , \quad \text{all } a$$

Problem 2.5

The system shown below contains two linear shift-invariant subsystems with unit sample responses $h_1(n)$ and $h_2(n)$, in cascade.

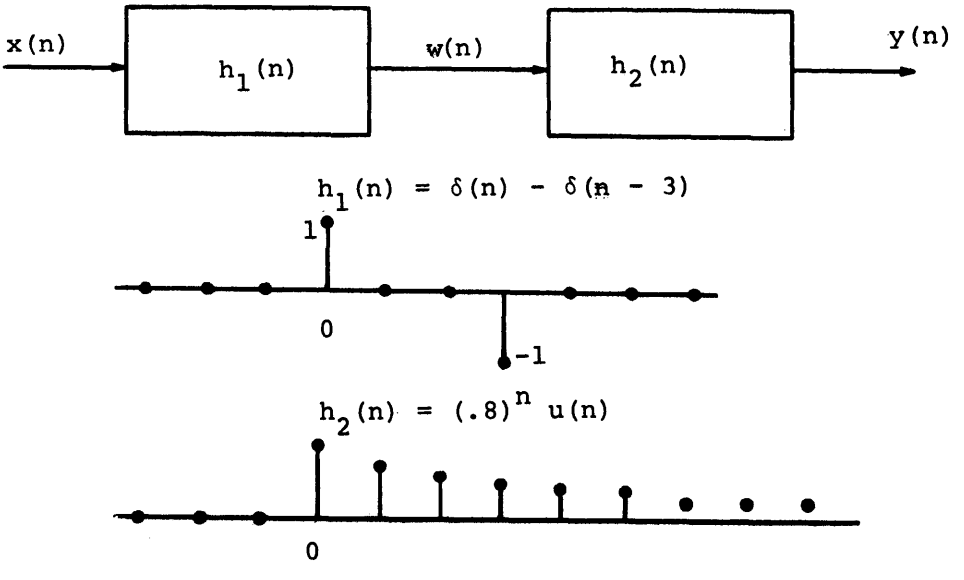


Figure P2.5-1

(a) Let $x(n) = u(n)$. Find $y_a(n)$ by first convolving $x(n)$ with $h_1(n)$ and then convolving the result with $h_2(n)$ i.e.

$$y_a(n) = [x(n) * h_1(n)] * h_2(n)$$

(b) Again let $x(n) = u(n)$. Find $y_b(n)$ by convolving $x(n)$ with the result of the convolution of $h_1(n)$ and $h_2(n)$ i.e.

$$y_b(n) = x(n) * [h_1(n) * h_2(n)] \quad .$$

Your results for parts (a) and (b) should be identical, illustrating the associative property of convolution.

Problem 2.6*

If the output of a system is the input multiplied by a complex constant then that input function is called an eigenfunction of the system.

(a) Show that the function $x(n) = z^n$, where z is a complex constant, is an eigenfunction of a linear shift-invariant discrete-time system.

(b) By constructing a counterexample, show that $z^n u(n)$ is not an eigenfunction of a linear shift-invariant discrete-time system.

* Asterisk indicates optional problem.

DISCRETE-TIME SIGNALS AND SYSTEMS, PART 1

Solution 2.1

$x(n)$ is periodic if $x(n) = x(n + N)$ for some integer value of N. For the sequence in (a),

$$x(n + N) = A\cos\left(\frac{3\pi}{7} n + \frac{3\pi}{7} N - \frac{\pi}{8}\right)$$

$x(n + N) = x(n)$ if $\frac{3\pi}{7} N$ is an integer multiple of 2π. The smallest value of N for which this is true is N = 14. Therefore the sequence in (a) is periodic with period 14.

For the sequence in (b),

$$x(n + N) = e^{j\left(\frac{n}{8} + \frac{N}{8} - \pi\right)}$$

$$= e^{j\left(\frac{n}{8} - \pi\right)} e^{j\frac{N}{8}} = x(n)\, e^{j\frac{N}{8}}$$

The factor $e^{j\frac{N}{8}}$ is unity for (N/8) an integer multiple of 2π. This requires that

$$\frac{N}{8} = 2\pi R$$

where N and R are both integers. This is not possible since π is an irrational number. Therefore this sequence is not periodic.

Solution 2.2

$$x(n) = -2\delta(n + 3) - \delta(n) + 3\delta(n - 1) + 2\delta(n - 3)$$

Solution 2.3

Each of the systems given can be tested against the definitions of linearity and time invariance. For example, for

(a), $T[x_1(n)] = 2x_1(n) + 3$

$T[x_2(n)] = 2x_2(n) + 3$

Since $T[ax_1(n) + bx_2(n)] = 2[ax_1(n) + bx_2(n)] + 3$

and $aT[x_1(n)] + bT[x_2(n)] = 2ax_1(n) + 2bx_2(n) + 3(a + b)$

The system is not linear. The system is, however, shift-invariant since $T[x(n-n_o)] = 2x(n-n_o) + 3 = y(n-n_o)$.

In a similar manner we can show that:

(b) is linear but not shift-invariant

(c) is not linear but is shift-invariant

(d) is linear and shift-invariant

Solution 2.4

To determine y(n) we evaluate the convolution sum eq. (2.39) of the text. For part (a), the sequences x(k) and h(n - k) are indicated below as functions of k:

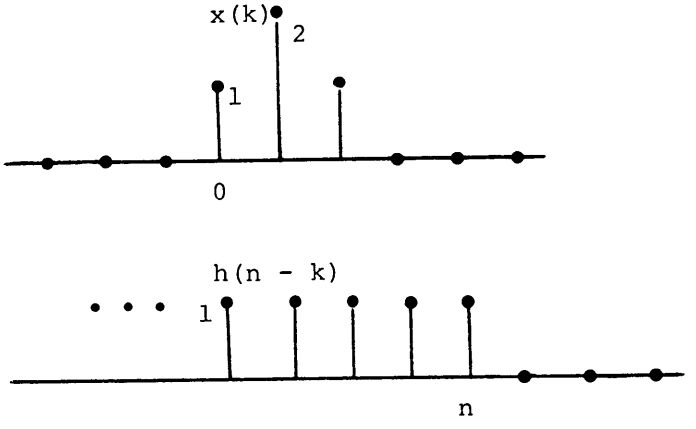


Figure S2.4-1

Since h(n - k) is zero for k > n, and is unity for k < n,

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k)\, h(n-k) = \sum_{k=-\infty}^{n} x(k)$$

as sketched below:

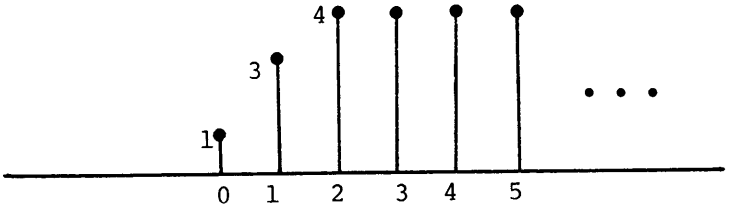


Figure S2.4-2

Part (b) can likewise be done graphically. Alternatively since $h(n) = \delta(n + 2)$,

$$y(n) = \sum_{k=-\infty}^{+\infty} h(k)\, x(n - k)$$

$$= \sum_{k=-\infty}^{+\infty} \delta(k + 2)\, x(n - k)$$

Since $\delta(k + 2) = 0$ except for $k = -2$, and is unity for $k = -2$ $y(n) = x(n + 2)$.

For part (c) $x(k)$ and $h(n - k)$ are as sketched below:

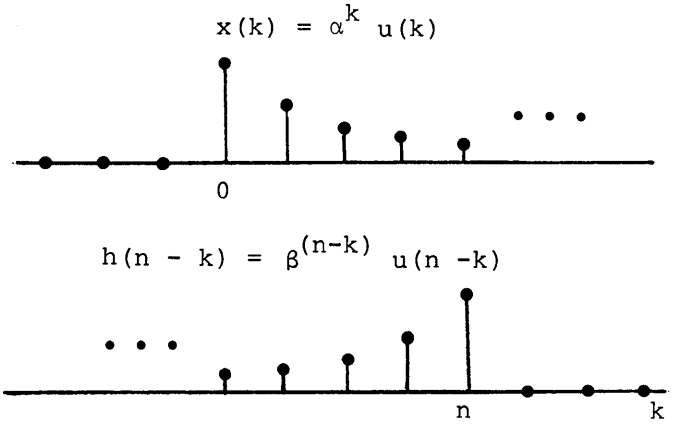


Figure S2.4-3

Graphically we see that for $n < 0$, $x(k)\, h(n - k)$ is zero and consequently $y(n) = 0$, $n < 0$. For $n \geq 0$

$$y(n) = \sum_{k=0}^{n} \alpha^k\, \beta^{n-k} = \beta^n \sum_{k=0}^{n} (\alpha/\beta)^n$$

$$= \beta^n\, \frac{1-(\alpha/\beta)^{n+1}}{1-(\alpha/\beta)} = \frac{\beta^{n+1}-\alpha^{n+1}}{\beta-\alpha}$$

Consequently for all n,

$$y(n) = \left[\frac{\beta^{n+1}-\alpha^{n+1}}{\beta-\alpha}\right] u(n)$$ which is a decaying exponential for $n \geq 0$.

The answer for part (d) is:

$$y(n) = x(n - 2) - x(n - 3) = \delta(n - 2)$$

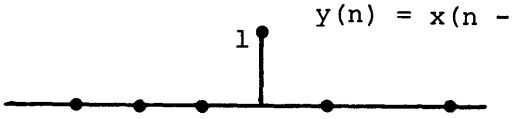
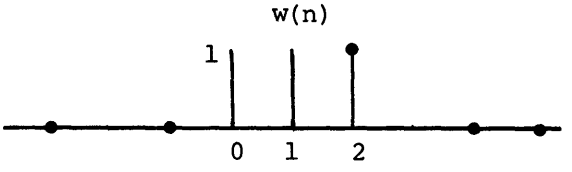


Figure S2.4-4

Solution 2.5*

$x(n) * h_1(n) = x(n) * [\delta(n) - \delta(n - 3)] = x(n) - x(n - 3)$.

Therefore with x(n) as a unit step, $x(n) * h_1(n)$ is:



Convolving w(n) graphically with $h_2(n)$

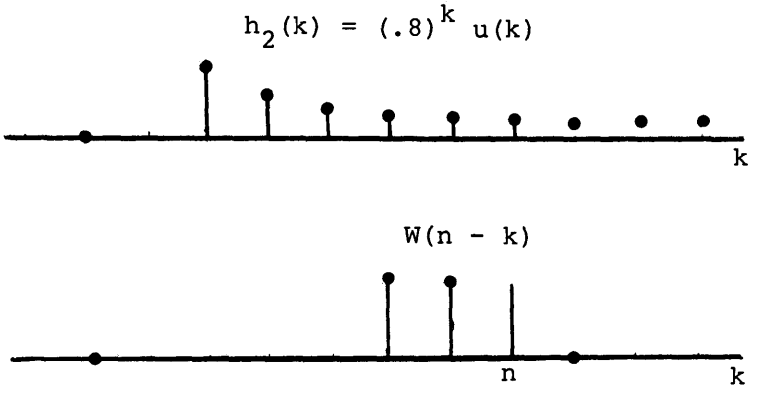


Figure S2.5-1

For $n < 0$ $h_2(k)\ w(n - k) = 0$

For $n = 0$ $y(n) = 1$

For $n = 1$ $y(n) = 1 + (.8)$

For $n \geq 2$ $y(n) = (.8)^{n-2} + (.8)^{n-1} + (.8)^n$

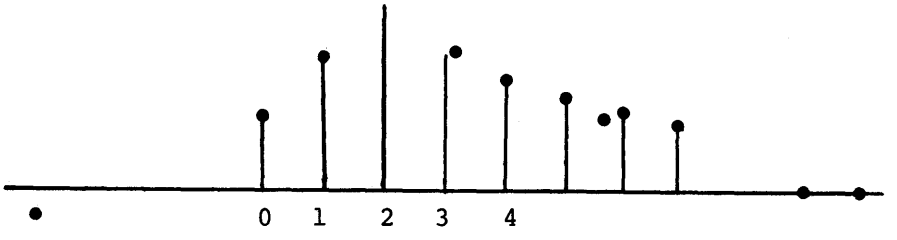


Figure S2.5-2

(b) The convolution of $h_1(n)$ and $h_2(n)$ is:

$$h(n) = h_1(n) * h_2(n) = (.8)^n u(n) - (.8)^{(n-3)} u(n-3)$$

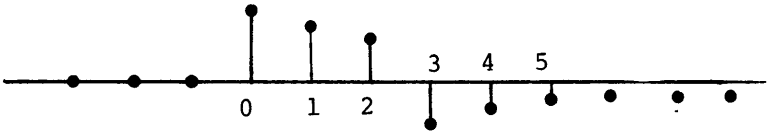


Figure S2.5-3

The convolution of this result with a unit step results in

$$y(n) = \sum_{k=-\infty}^{n} h(k)$$

or

$$y(n) = 0 \quad n < 0$$

$$y(0) = 1$$

$$y(1) = 1 + .8$$

$$y(2) = 1 + (.8) + (.8)^2$$

$$y(3) = 1 + .8 + (.8)^2 + (.8)^3 - 1 = .8 + (.8)^2 + (.8)^3$$

$$y(4) = 1 + .8 + (.8)^2 + [(.8)^3 - 1] + [(.8)^4 - .8]$$

$$= (.8)^2 + (.8)^3 + (.8)^4$$

etc.

Solution 2.6*

The fact that $x(n) = z^n$ is an eigenfunction follows from the convolution sum. Specifically

$$y(n) = \sum_{k=-\infty}^{+\infty} h(k)\, x(n-k) = \sum_{k=-\infty}^{+\infty} h(k) z^{(n-k)}$$

$$= z^n \sum_{k=-\infty}^{+\infty} h(k) z^{-k} \qquad \text{(S2.6-1)}$$

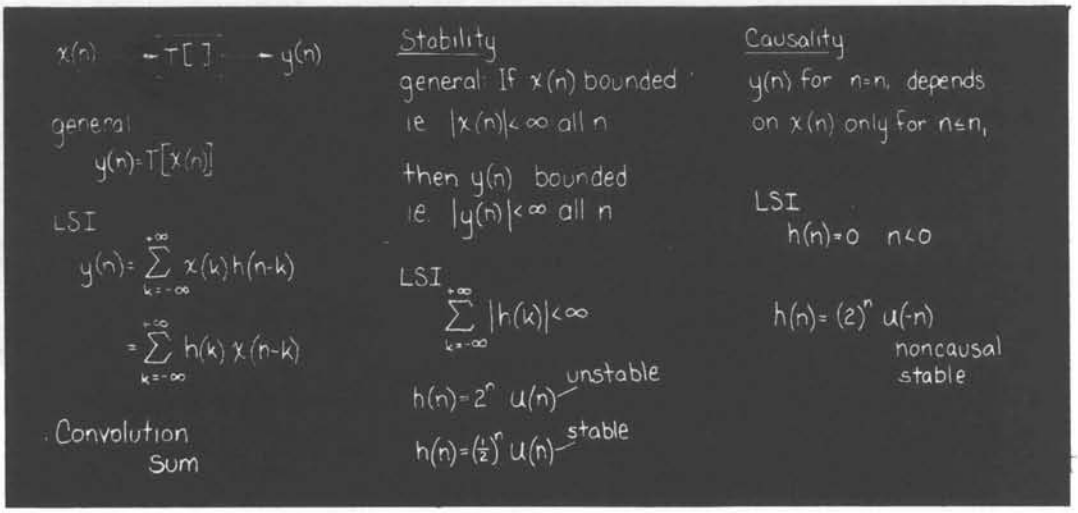
Since the summation in the equation (S2.6-1) does not depend on n, it is simply a constant for any given z.

While the complex exponential z^n is an eigenfunction of any linear shift-invariant system, $z^n u(n)$ is not. For example, let $h(n) = \delta(n-1)$. Then with $x(n) = z^n u(n)$, $y(n) = z^{n-1} u(n-1)$, which is not a complex constant times $x(n)$.

DISCRETE-TIME SIGNALS AND SYSTEMS, PART 2

1. Lecture 3 - 50 minutes

a.



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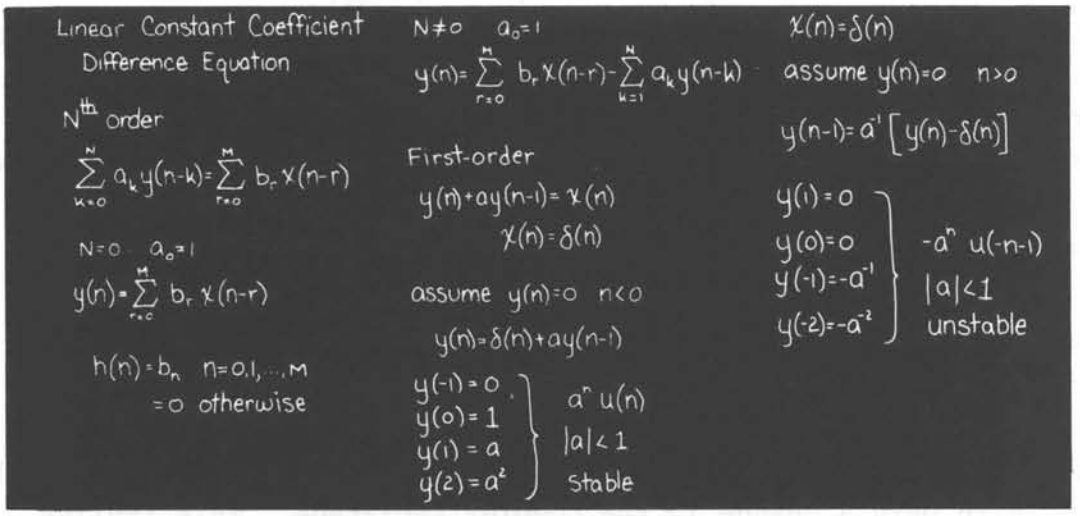

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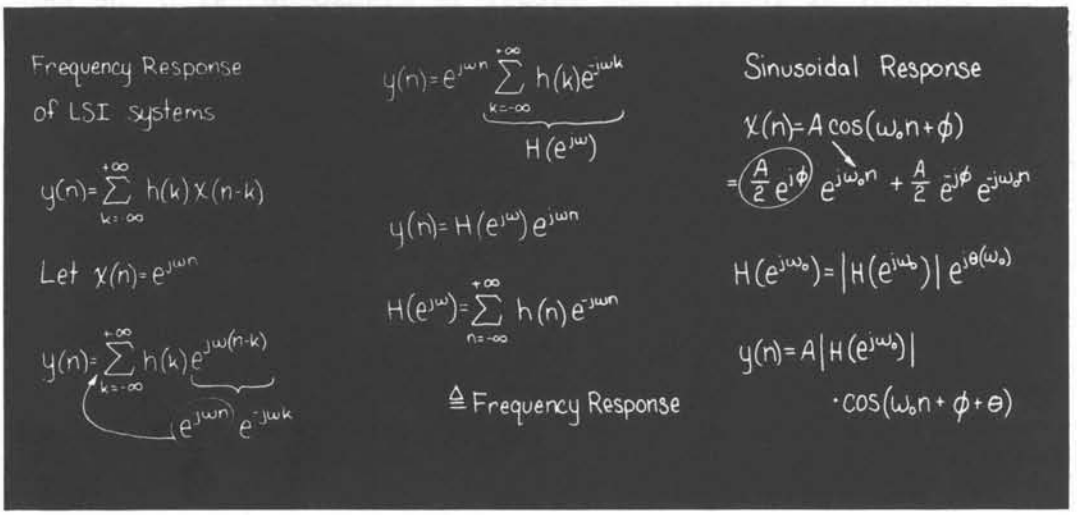

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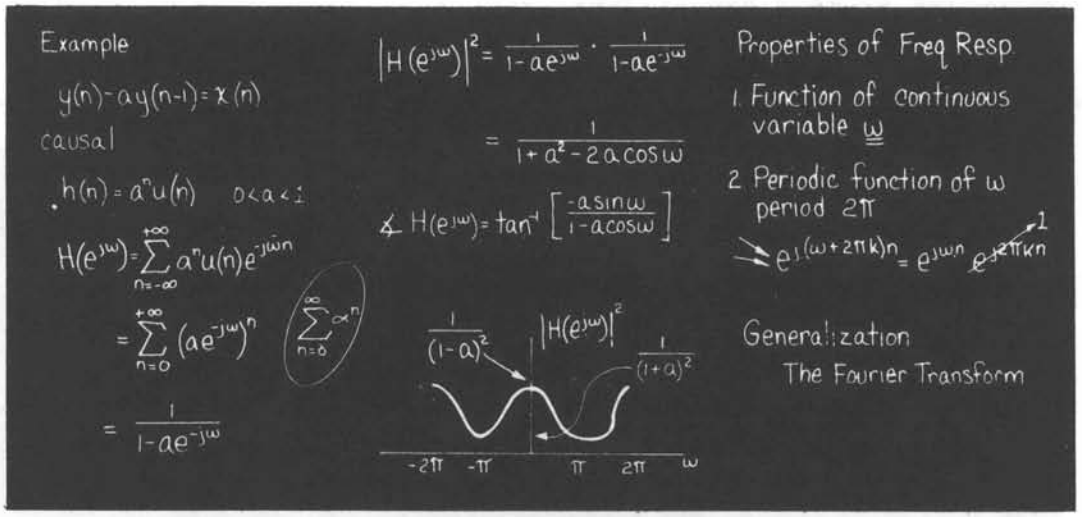


2. Comments

This lecture continues the discussion of discrete-time systems. We consider, in particular, the definitions for stability and causality, first for discrete-time systems in general, and then for the particular class of linear shift-invariant systems. We also introduce in this lecture the class of linear shift-invariant systems characterized by linear constant-coefficient difference equations.

As the final topic in this lecture we discuss the notion of the frequency response of LSI systems. The description of LSI systems in terms of the frequency response is based on the fact that complex exponentials are eigenfunctions of this class of systems, i.e. the output due to a complex exponential input is of the same form as the input with only a change in the complex amplitude. This implies that for sinusoidal inputs, the output is sinusoidal with the same frequency but with a change in amplitude and phase. Just as with continuous-time systems, the frequency domain representation of discrete-time LSI systems is one of the cornerstones of digital signal processing.

3. Reading

Text: Section 2.4 starting after eq.(2.51) (page 28) sections 2.5 and 2.6.

4. Problems

Problem 3.1

Stability of a system was defined in the lecture and in section 2.2.5 of the text, and it was shown that for an LSI system a necessary and sufficient condition for stability is that h(n) be absolutely summable. Show that each of the unit sample responses listed below corresponds to a stable system

(i) $h(n) = \delta(n + 2)$

(ii) $h(n) = (\frac{1}{2})^n u(n)$

(iii) $h(n) = 2^n u(-n)$

Problem 3.2

Causality of a system was defined in the lecture and in section 1.3 of the text. In this problem we wish to show that for an LSI system a necessary and sufficient condition for causality is that the unit sample response be zero for $n < 0$.

(a) Show from the convolution sum that if $h(n) = 0$ for $n < 0$ then the system must be causal, i.e. that $y(n_0)$ depends only on $x(n)$ for $n \leq n_0$ where n_0 is arbitrary. This then shows that a sufficient condition for causality of an LSI system is that $h(n) = 0$, $n < 0$.

(b) Now, suppose that $h(n)$ is <u>not</u> zero for $n < 0$. Argue that the system <u>cannot</u> be causal. This can be done simply by showing that there is at least one input for which the output "anticipates" the input i.e. for which $y(n_0)$ depends on values of $x(n)$ for $n > n_0$. This then establishes that a necessary condition for causality of an LSI system is that $h(n) = 0$, $n < 0$.

Problem 3.3

Determine whether or not each of the following systems is stable and/or causal:

(a) $y(n) = g(n)x(n)$ where $g(n)$ is bounded

(b) $y(n) = \sum_{k=n_0}^{n} x(k)$

(c) $y(n) = x(n - n_0)$

Problem 3.4

Consider a causal system for which the input x(n) and output y(n) are related by the linear constant coefficient difference equation

$$y(n) - \frac{1}{2} y(n-1) = x(n) + \frac{1}{2} x(n-1)$$

(a) Determine the unit sample response of the system.

(b) Using your result in (a) and the convolution sum determine the response to the input

$$x(n) = e^{j\omega n}$$

(c) Determine the frequency response of the system;

(d) Determine the response of the system to the input

$$x(n) = \cos(\frac{\pi}{2} n + \frac{\pi}{4})$$

Problem 3.5*

Consider a system with input x(n) and output y(n). The input-output relation for the system is defined by the following two properties:

(1) $y(n) - ay(n-1) = x(n)$.

(2) $y(0) = 1$.

(a) Determine whether the system is shift-invariant.

(b) Determine whether the system is linear.

Problem 3.6*

Consider a discrete-time system with input x(n) and output y(n). The system transformation $y(n) = F[x(n)]$ is arbitrary and may be nonlinear and time-varying. The only known property of the system is that it is well defined, i.e. the output for any given input is unique. Suppose that the input x(n) is chosen as $x(n) = e^{j\omega n}$ and some parameter P of the output is measured (e.g. the maximum amplitude). P will in general be a function of ω. Let us consider P for different excitation frequencies. Show that P is periodic in ω and determine the period.

DISCRETE-TIME SIGNALS AND SYSTEMS, PART 2

Solution 3.1

To correspond to a stable system, the unit sample response must be absolutely summable. For each system

$\sum_{n=-\infty}^{+\infty} |h(n)|$ is given by:

(i) 1

(ii) $\sum_{n=0}^{\infty} (\frac{1}{2})^n = 2$

(iii) $\sum_{n=-\infty}^{0} 2^n = \sum_{n=0}^{\infty} (\frac{1}{2})^n = 2$

Solution 3.2

(a) Consider the convolution sum in the form

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k)\, h(n - k)$$

If $h(n) = 0$, $n < 0$, then $h(n - k) = 0$, $k > n$ and consequently

$$y(n) = \sum_{k=-\infty}^{n} x(k)\, h(n - k) \quad .$$

Thus $y(n_0)$ is given by

$$y(n_0) = \sum_{k=-\infty}^{n_0} x(k)\, h(n_0 - k)$$

and hence depends only on values of $x(k)$ for $k \leq n_0$.

(b) If $h(n)$ is not zero for $n < 0$, then an example of an input for which the output anticipates the input is, of course, a unit sample. Alternatively, consider the convolution sum

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k)\, h(n - k)$$

or,

$$y(n) = \sum_{k=-\infty}^{n} x(k)\, h(n - k) + \sum_{k=n+1}^{\infty} x(k)\, h(n - k)$$

If $h(n)$ is not zero for $n < 0$ then at least one of the terms in the summation $\sum_{k=n+1}^{+\infty} x(k)\, h(n-k)$ will be non-zero, i.e. $y(n)$ will depend on at least one value in $x(k)$ for $k > n$.

Solution 3.3

Not all of these systems correspond to LSI systems. For those that don't we can examine stability and causality by referring back to the basic definitions.

(a) Clearly causal. Also stable since

$$|y(n)| = |g(n)|\,|x(n)|$$

Thus, if $x(n)$ is bounded then $y(n)$ will be bounded

(b) Not causal, since for $n < n_0$, $y(n)$ will depend on values in $x(k)$ for $k > n$. Also, not stable, since if $x(n) = 1$ for all n then $y(n) = (n - n_0 + 1)$ which is unbounded.

(c) Clearly stable. It is causal if $n_0 \geq 0$ and not causal otherwise.

Solution 3.4

(a) Rewriting the difference equation, with $x(n) = \delta(n)$ and $h(n)$ denoting the unit-sample response

$$h(n) = \frac{1}{2} h(n-1) + \delta(n) + \frac{1}{2} \delta(n-1) \quad .$$

Since the system is causal, $h(n)$ is zero for $n < 0$. For $n \geq 0$,

$$h(0) = \frac{1}{2} h(-1) + \delta(0) + \frac{1}{2} \delta(-1) = 1$$

$$h(1) = \frac{1}{2} h(0) + \delta(1) + \frac{1}{2} \delta(0) = 1$$

$$h(2) = \frac{1}{2} h(1) + \delta(2) + \frac{1}{2} \delta(1) = \frac{1}{2}$$

$$h(n) = 2\left(\frac{1}{2}\right)^n \quad n \geq 1$$

$h(n)$ can also be expressed as

$$h(n) = \left(\frac{1}{2}\right)^n [u(n) + u(n-1)] \quad .$$

(b) Substituting x(n) and h(n) into the convolution sum we obtain

$$y(n) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k \left[u(k) + u(k-1)\right] e^{j\omega(n-k)}$$

$$= e^{j\omega n} \left[\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k e^{-j\omega k} + \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k e^{-j\omega k}\right]$$

$$= e^{j\omega n} \left[\frac{1 + \frac{1}{2} e^{-j\omega}}{1 - \frac{1}{2} e^{-j\omega}}\right]$$

(c) The frequency response of the system is the complex amplitude of the response with an excitation $e^{j\omega n}$. Thus, from part (a)

$$H(e^{j\omega}) = \frac{1 + \frac{1}{2} e^{-j\omega}}{1 - \frac{1}{2} e^{-j\omega}}$$

(d) With $H(e^{j\omega})$ expressed in polar form as $|H(e^{j\omega})|\ e^{j\Theta(\omega)}$, the response to the specified input is

$$y(n) = \left|H(e^{j\frac{\pi}{2}})\right| \cos\left(\frac{\pi n}{2} + \frac{\pi}{4} + \Theta\left(\frac{\pi}{2}\right)\right)$$

From part (c),

$$\left|H(e^{j\frac{\pi}{2}})\right| = 1$$

$$\Theta\left(\frac{\pi}{2}\right) = -2 \tan^{-1}\left(\frac{1}{2}\right)$$

Solution 3.5*

(a) If $x(n) = \delta(n)$ then the system response is $y(n) = a^n u(n)$. If $x(n) = \delta(n-1)$ then for $n > 0$ the system response will be $(a^n + a^{n-1})$. Therefore the system is not shift-invariant.

(b) The system is not linear, as can be demonstrated in a variety of ways. For example, a linear system has the property that $T[a\ x(n)] = aT[x(n)]$. Therefore if the input is doubled (for example) the output must double at each value of n, which for this system cannot happen at n = 0.

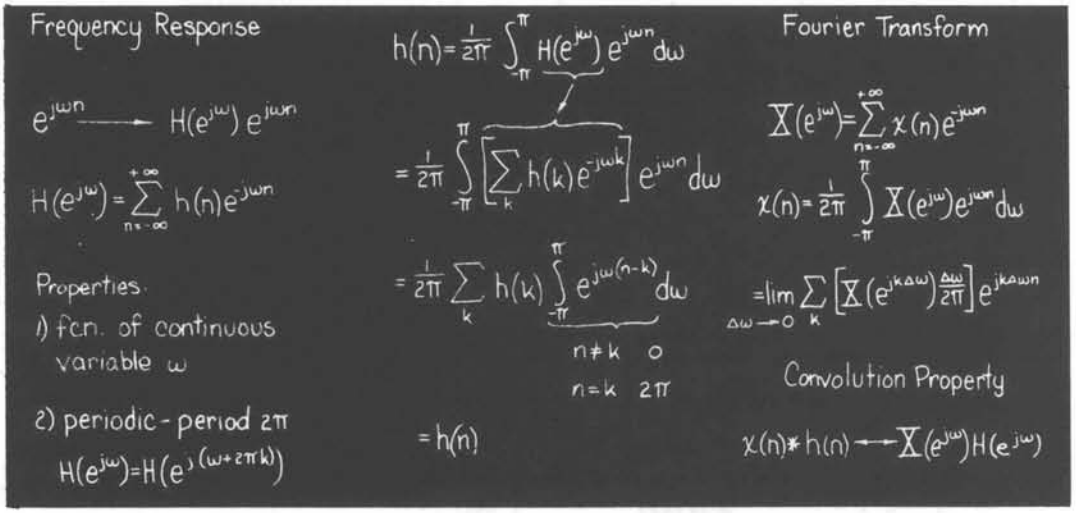
Solution 3.6*

The system excitation is $x(n) = e^{j\omega n}$. With ω replaced by $(\omega + 2\pi)$ the excitation becomes $e^{j(\omega+2\pi)n} = e^{j\omega n}\, e^{j2\pi n} = e^{j\omega n}$ and hence the input is periodic in ω. This means, in essence that in considering complex exponential inputs, a variation in ω over a range of 2π generates all of the distinguishable discrete-time complex exponentials. When ω varies outside that range we simply see the same ones over again. For the system described in this problem, the output for a given input is unique. Thus since $e^{j\omega n}$ and $e^{j(\omega+2\pi)n}$ are identical input signals their outputs must be identical, and consequently the output parameter P will be periodic with period 2π.

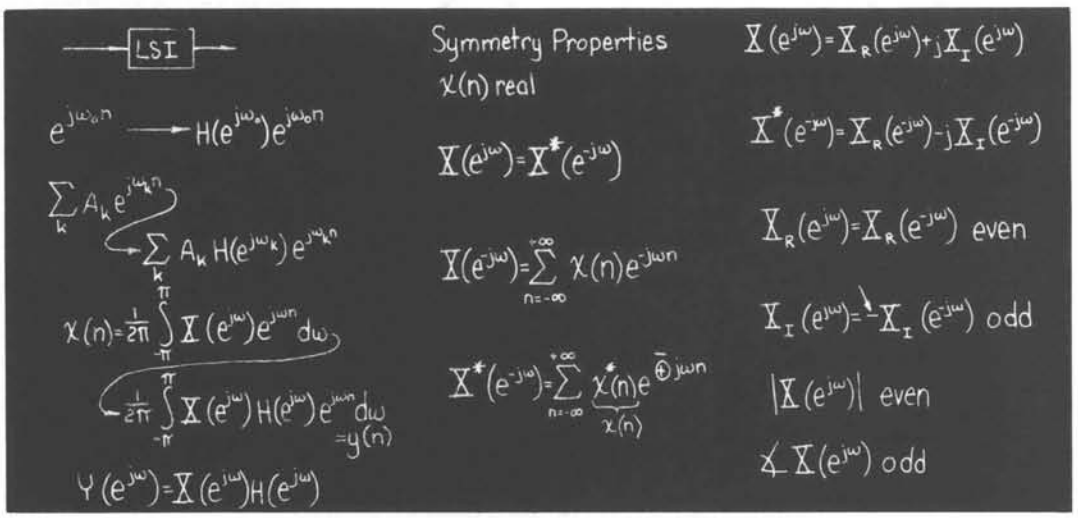
THE DISCRETE-TIME FOURIER TRANSFORM

1. Lecture 4 - 44 minutes

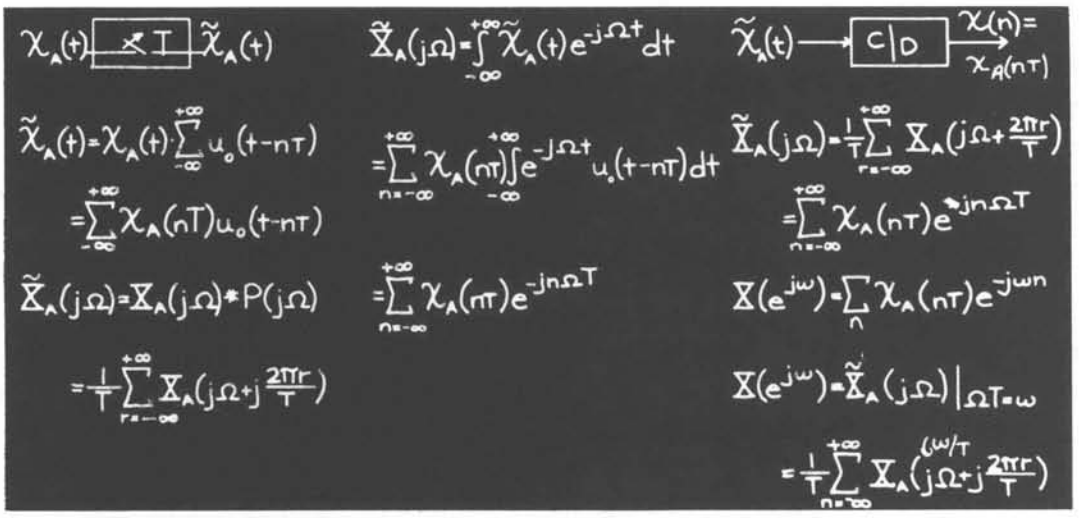
a.

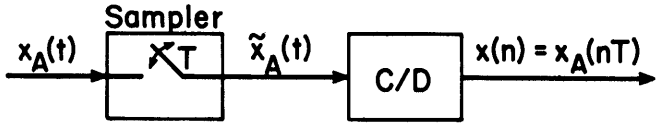

Frequency Response
$e^{j\omega n} \longrightarrow H(e^{j\omega}) e^{j\omega n}$
$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h(n) e^{-j\omega n}$
Properties.
1) fcn. of continuous variable ω
2) periodic - period 2π
$H(e^{j\omega}) = H(e^{j(\omega + 2\pi k)})$
$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega$
$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\sum_k h(k) e^{-j\omega k} \right] e^{j\omega n} d\omega$
$= \frac{1}{2\pi} \sum_k h(k) \int_{-\pi}^{\pi} e^{j\omega(n-k)} d\omega$
$n \neq k \quad 0$
$n = k \quad 2\pi$
$= h(n)$
Fourier Transform
$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$
$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$
$= \lim_{\Delta\omega \to 0} \sum_k \left[X(e^{jk\Delta\omega}) \frac{\Delta\omega}{2\pi} \right] e^{jk\Delta\omega n}$
Convolution Property
$x(n) * h(n) \longrightarrow X(e^{j\omega}) H(e^{j\omega})$

b.

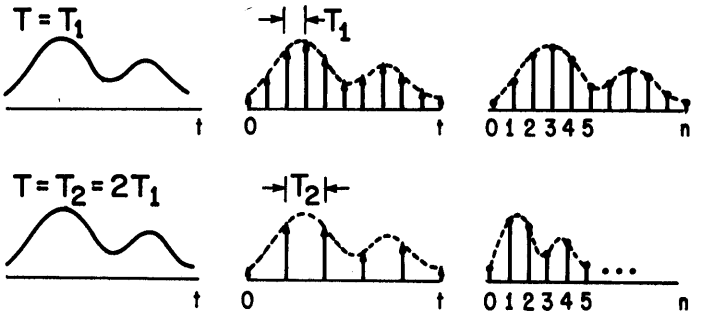

LSI
$e^{j\omega_0 n} \longrightarrow H(e^{j\omega_0}) e^{j\omega_0 n}$
$\sum_k A_k e^{j\omega_k n} \longrightarrow \sum_k A_k H(e^{j\omega_k}) e^{j\omega_k n}$
$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$
$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) H(e^{j\omega}) e^{j\omega n} d\omega = y(n)$
$Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$
Symmetry Properties
$x(n)$ real
$X(e^{j\omega}) = X^*(e^{-j\omega})$
$X(e^{-j\omega}) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$
$X^*(e^{-j\omega}) = \sum_{n=-\infty}^{+\infty} \underbrace{x^*(n)}_{x(n)} e^{\oplus j\omega n}$
$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})$
$X^*(e^{-j\omega}) = X_R(e^{-j\omega}) - jX_I(e^{-j\omega})$
$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ even
$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ odd
$|X(e^{j\omega})|$ even
$\measuredangle X(e^{j\omega})$ odd

c.

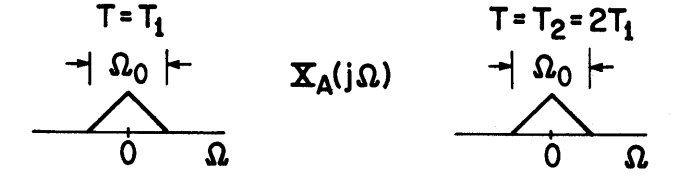

$x_A(t)$ — C/T — $\tilde{x}_A(t)$
$\tilde{x}_A(t) = x_A(t) \cdot \sum_{n=-\infty}^{+\infty} u_0(t - nT)$
$= \sum_{n=-\infty}^{+\infty} x_A(nT) u_0(t - nT)$
$\tilde{X}_A(j\Omega) = X_A(j\Omega) * P(j\Omega)$
$= \frac{1}{T} \sum_{r=-\infty}^{+\infty} X_A(j\Omega + j\frac{2\pi r}{T})$
$\tilde{X}_A(j\Omega) = \int_{-\infty}^{+\infty} \tilde{x}_A(t) e^{-j\Omega t} dt$
$= \sum_{n=-\infty}^{+\infty} x_A(nT) \int_{-\infty}^{+\infty} e^{-j\Omega t} u_0(t - nT) dt$
$= \sum_{n=-\infty}^{+\infty} x_A(nT) e^{-jn\Omega T}$
$\tilde{x}_A(t)$ — C/D — $x(n) = x_A(nT)$
$\tilde{X}_A(j\Omega) = \frac{1}{T} \sum_{r=-\infty}^{+\infty} X_A(j\Omega + \frac{2\pi r}{T})$
$= \sum_{n=-\infty}^{+\infty} x_A(nT) e^{+jn\Omega T}$
$X(e^{j\omega}) = \sum_n x_A(nT) e^{-j\omega n}$
$X(e^{j\omega}) = \tilde{X}_A(j\Omega)\big|_{\Omega T = \omega}$
$= \frac{1}{T} \sum_{n=-\infty}^{+\infty} X_A(j\frac{\omega}{T} + j\frac{2\pi r}{T})$



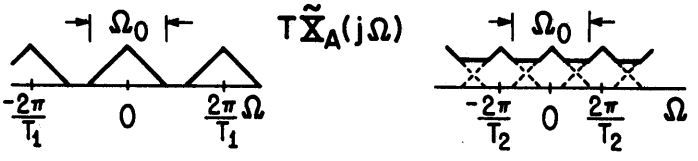
Relationship in the time domain for a continous-time signal, its samples, and the resulting sequence

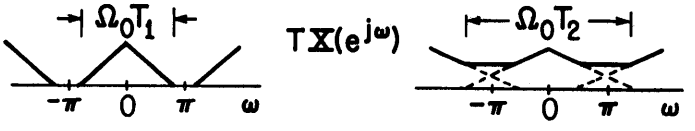


d.



Relationship among the Fourier transforms of a continous-time signal, its samples, and the resulting sequence.





e.

2. Comments

In the previous lecture we developed the notion of the frequency response of LSI systems and discussed the determination of the frequency response from the unit sample response. We begin this lecture by developing an inverse relation by interpreting the expression for the frequency response as a Fourier series expansion, the coefficients for which are the unit sample response values. We then consider the generalization of the frequency response representation of LSI systems to the Fourier transform representation of sequences. Throughout this set of lectures we will exploit various Fourier transform properties. Many of these properties are derived in a similar manner and it is useful to understand the style in which they are

derived rather than simply memorizing the properties. In the lecture the convolution property and some symmetry properties were discussed. A number of other important properties are presented in the text and should be carefully reviewed.

The lecture concludes with a discussion of the relationships between continuous-time and discrete-time Fourier transforms. In particular you should be aware from your background in continuous-time linear system theory of the form of the Fourier transform of a sampled time function. Converting a sampled time function to a sequence introduces in essence a "time" normalization since the spacing of sequence values is always interpreted to be unity. Equivalently this time normalization corresponds in the Fourier domain to a frequency normalization.

3. Reading

Text: Review section 2.6 (page 39). Read sections 2.7, 2.8, 2.9 and sections 3.1 through 3.5.

4. Problems

Problem 4.1

Determine the Fourier transform of each of the sequences below:

(a)

$x(n) = \delta(n - 3)$

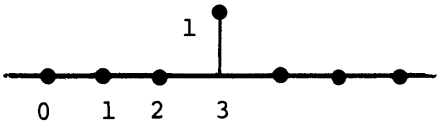


Figure P4.1-1

(b) $x(n) = \frac{1}{2}\delta(n + 1) + \delta(n) + \frac{1}{2}\delta(n - 1)$

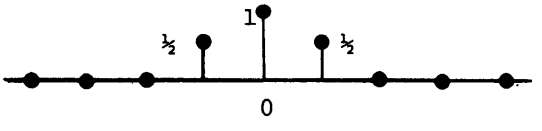


Figure P4.1-2

(c) $x(n) = a^n u(n) \quad 0<a<1$

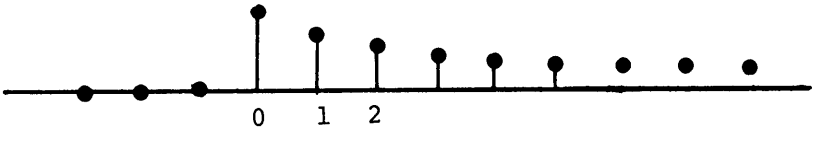


Figure P4.1-3

(d)

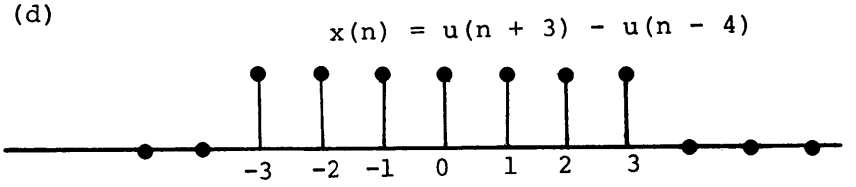


Figure P4.1-4

Problem 4.2

(a) Consider a linear shift-invariant system with unit-sample response $h(n) = \alpha^n u(n)$, where α is real and $0 < \alpha < 1$. If the input is $x(n) = \beta^n u(n)$, $0 < |\beta| < 1$, determine the output $y(n)$ in the form $y(n) = (k_1\alpha^n + k_2\beta^n) u(n)$ by explicitly evaluating the convolution sum.

(b) By explicitly evaluating the transforms $X(e^{j\omega})$, $H(e^{j\omega})$ and $Y(e^{j\omega})$ corresponding to $x(n)$, $h(n)$, and $y(n)$ specified in part (a), show that $Y(e^{j\omega}) = H(e^{j\omega})\ X(e^{j\omega})$

Problem 4.3

Let $x(n)$ and $X(e^{j\omega})$ represent a sequence and its transform. Do not assume that $x(n)$ is real or that $x(n)$ is zero for $n < 0$. Determine in terms of $X(e^{j\omega})$ the transform of each of the following:

(a) $k\ x(n)$

(b) $x(n - n_0)$ where n_0 is an integer

(c) $n\ x(n)$

Problem 4.4

Let $h_a(t)$ denote the impulse response of a linear time-invariant continuous-time filter and $h_d(n)$ the unit-sample response of a linear shift-invariant discrete-time filter. $h_a(t)$ is given by

$$h_a(t) = \begin{cases} ae^{-at} & t \geq 0 \qquad a > 0 \\ 0 & t < 0 \end{cases}$$

(a) Determine the analog filter frequency response and sketch its magnitude.

(b) If $h_d(n) = c\ h_a(nT)$, determine the digital filter frequency response and determine the value of c so that the digital filter has unity gain at $\omega = 0$. Sketch the magnitude of $H_d(e^{j\omega})$.

Problem 4.5

One context in which digital filters are frequently used is in filtering bandlimited analog data as shown below:

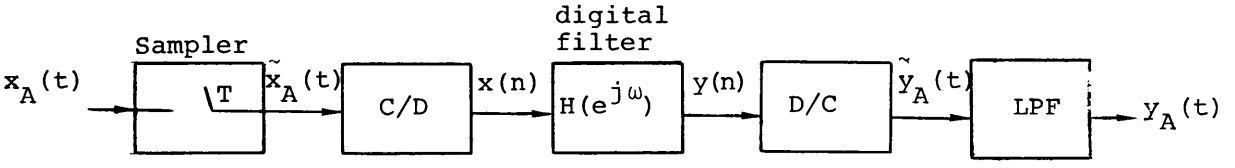


Figure P4.5-1

Assume that the sampling period T is sufficiently small to prevent aliasing. The system labelled D/C is a discrete to continuous time converter, i.e. the inverse of a continuous to discrete time converter. The system labelled LPF is an ideal lowpass filter with a gain of T in the passband and cutoff frequency of $\frac{\pi}{T}$ rad/sec. The overall system is equivalent to a continous-time filter. Indicated below are two choices for T and the digital filter frequency response $H(e^{j\omega})$. For each of these, sketch the frequency response of the overall continuous-time system.

(a)

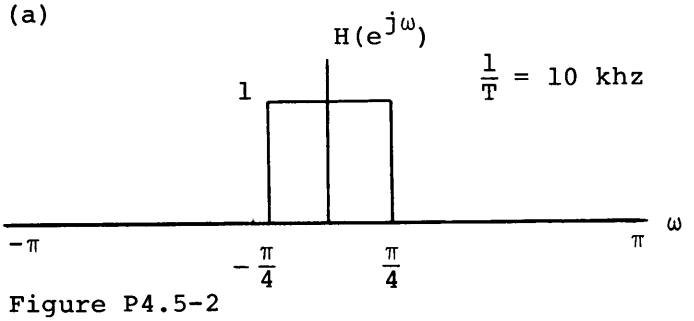


Figure P4.5-2

(b)

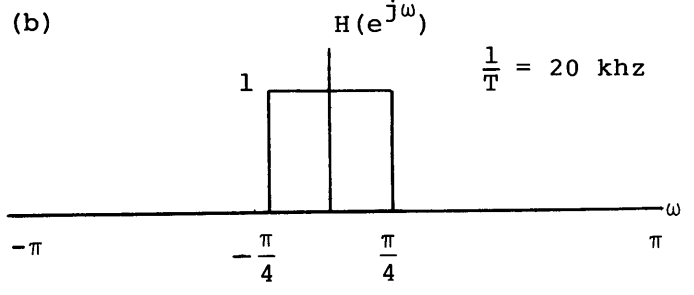


Figure P4.5-3

Problem 4.6*

In Sec. 1.6 of the text, a number of symmetry properties of the Fourier transform are stated. All these properties follow in a relatively straightforward way from the transform pair. Following is a list of some of the

properties stated. Prove that each is true. In carrying out the proof you may use the definition of the transform pair as given in Eqs. (1.19) and (1.20) of the text and any previous property in the list. For example, in proving property 3 you may use properties 1 and 2.

Sequence	Fourier Transform
1. $x^*(n)$	$X^*(e^{-j\omega})$
2. $x^*(-n)$	$X^*(e^{j\omega})$
3. Re $x(n)$	$X_e(e^{j\omega})$
4. j Im $[x(n)]$	$X_0(e^{j\omega})$
5. $x_e(n)$	Re $[X(e^{j\omega})]$
6. $x_0(n)$	j Im $[X(e^{j\omega})]$

Problem 4.7*

f(n) and g(n) are causal and stable sequences with Fourier transforms $F(e^{j\omega})$ and $G(e^{j\omega})$, respectively. Show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{j\omega})\ G(e^{j\omega})\ d\omega = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{j\omega})\ d\omega\right\}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})\ d\omega\right\}$$

Problem 4.8

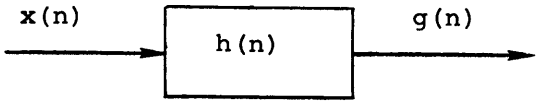
In the design of either analog or digital filters, we often approximate a specified magnitude characteristic, without particular regard to the phase. For example, standard design techniques for lowpass and bandpass filters are derived from consideration of the magnitude characteristics only.

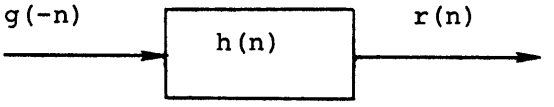
In many filtering problems, one would ideally like the phase characteristics to be zero or linear. For causal filters it is impossible to have zero phase. However, for many digital filtering applications, it is not necessary that the unit-sample response of the filter be zero for $n < 0$ if the processing is not to be carried out in real time.

One technique commonly used in digital filtering when the data to be filtered are of finite duration and stored, for example, on a disc or magnetic tape, is to process the data forward and then backward through the same filter.

Let h(n) be the unit-sample response of a causal filter with an arbitrary phase characteristic. Assume that h(n) is real and denote its Fourier transform by $H(e^{j\omega})$. Let x(n) be the data we want to filter. The filtering operation is performed as follows.

(a) Method A:

i. 

ii. 

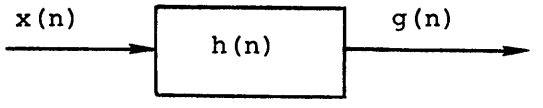
iii. $s(n) = r(-n)$

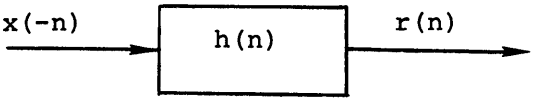
Figure P4.8-1

(1) Determine the overall unit-sample response $h_1(n)$ that relates $x(n)$ and $s(n)$, and show that it has a zero-phase characteristic.

(2) Determine $|H_1(e^{j\omega})|$ and express it in terms of $H(e^{j\omega})$ and $\arg[H(e^{j\omega})]$.

(b) Method B: Process $x(n)$ through the filter $h(n)$ to get $g(n)$. Also process $x(n)$ backward through $h(n)$ to get $r(n)$. The output $y(n)$ is then taken as the sum $g(n)$ plus $r(-n)$.

i. 

ii. 

iii $y(n) = g(n) + r(-n)$

Figure P4.8-2

This composite set of operations can be represented by a filter, with input $x(n)$, output $y(n)$, and unit-sample response $h_2(n)$.

(1) Show that the composite filter $h_2(n)$ has a zero-phase characteristic.
(2) Determine $|H_2(e^{j\omega})|$ and express it in terms of $|H(e^{j\omega})|$ and $\arg[H(e^{j\omega})]$.

c. Suppose that we are given a sequence of finite duration, on which we would like to perform a bandpass, zero-phase filtering operation. Furthermore, assume that we are given the bandpass filter $h(n)$, with frequency response as specified in Figure P4.8-3, which has the magnitude characteristic that we desire but linear phase. To achieve zero phase, we could use either method (A) or (B). Determine and sketch $|H_1(e^{j\omega})|$ and $|H_2(e^{j\omega})|$. From these results which method would you

use to achieve the desired bandpass filtering operation? Explain why. More generally, if h(n) has the desired magnitude but a nonlinear phase characteristic which method is preferable to achieve a zero phase characteristic?

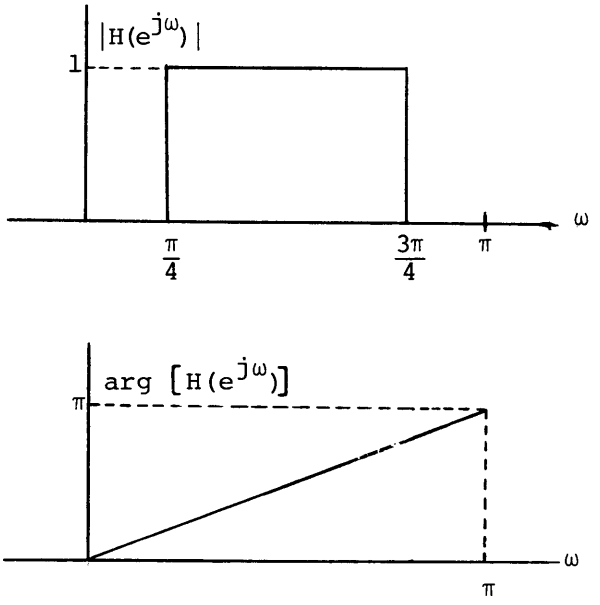


Figure P4.8-3

THE DISCRETE-TIME FOURIER TRANSFORM

Solution 4.1

The Fourier transform relation is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\ e^{-j\omega n} \qquad \text{thus:}$$

(a) $$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta(n-3)\ e^{-j\omega n} = e^{-j\omega 3}$$

(b) $$X(e^{j\omega}) = 1 + \frac{1}{2}e^{j\omega} + \frac{1}{2}e^{-j\omega} = 1 + \cos\omega$$

(c) $$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n\ e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

(d) $$X(e^{j\omega}) = \sum_{n=-3}^{+3} e^{-j\omega n} = e^{j3\omega}\sum_{n=0}^{6} e^{-j\omega n} = \frac{\sin(\frac{7\omega}{2})}{\sin(\frac{\omega}{2})}$$

Solution 4.2

(a) In problem 2.4(c) we determined that the convolution of $\alpha^n\ u(n)$ and $\beta^n\ u(n)$ was given by

$$y(n) = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha}\right] u(n)$$

$$y(n) = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha}\right] u(n)$$

$$= \left[\alpha^n \left(\frac{\alpha}{\alpha-\beta}\right) + \beta^n \left(\frac{-\beta}{\alpha-\beta}\right)\right] u(n)$$

thus, $k_1 = \frac{\alpha}{\alpha-\beta}$ and $k_2 = \frac{\beta}{\beta-\alpha}$

(b) From problem 4.1 (c) it follows that

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$$

and

$$X(e^{j\omega}) = \frac{1}{1 - \beta e^{-j\omega}}$$

The Fourier transform of y(n) as obtained in (a)

$$Y(e^{j\omega}) = \sum_{n=0}^{\infty} \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha} e^{-j\omega n}$$

$$= \frac{\beta}{\beta - \alpha} \sum_{n=0}^{\infty} \beta^n e^{-j\omega n} - \frac{\alpha}{\beta - \alpha} \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \frac{\beta}{\beta - \alpha} \frac{1}{1 - \beta e^{-j\omega}} - \frac{\alpha}{\beta - \alpha} \frac{1}{1 - \alpha e^{-j\omega}}$$

$$= \frac{1}{(1 - \beta e^{-j\omega})(1 - \alpha e^{-j\omega})}$$

Solution 4.3

(a) $X_a(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} kx(n)e^{-j\omega n} = k\sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n}$

$= k\ X(e^{j\omega})$

(b) $X_b(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n - n_0)\ e^{-j\omega n}$

Making the substitution of variables

$m = n - n_0$ or $n = m + n_0$

$$X_b(e^{j\omega}) = \sum_{m=-\infty}^{+\infty} x(m)\ e^{-j\omega(m+n_0)} = \sum_{m=-\infty}^{+\infty} e^{-j\omega n_0}\ x(m)\ e^{-j\omega m}$$

$$= e^{-j\omega n_0}\ X(e^{j\omega})$$

(c) The transform of x(n) is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\ e^{-j\omega n}$$

thus $$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{+\infty} (-jn)\ x(n)\ e^{-j\omega n}$$

or

$$j\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{+\infty} n\ x(n)\ e^{-j\omega n}$$

$$= X_c(e^{j\omega})$$

Solution 4.4

(a) $H_a(j\Omega) = \int_{-\infty}^{+\infty} h_a(t)\, e^{-j\Omega t}\, dt = \int_0^{\infty} ae^{-at}\, e^{-j\Omega t}\, dt = \frac{a}{j\Omega + a}$

$|H_a(j\Omega)|^2 = \frac{a^2}{\Omega^2 + a^2}$. Thus $|H(j\Omega)|$ is as sketched below:

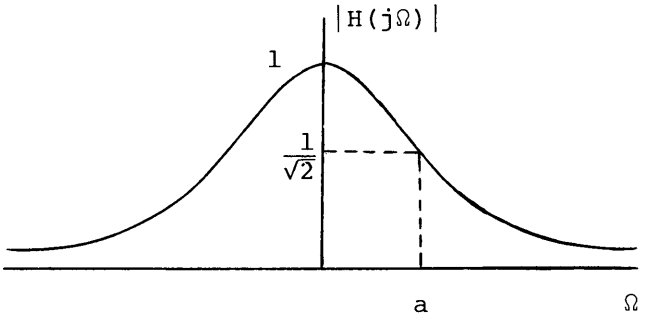


Figure S4.4-1

(b) $h_d(n) = cae^{-anT}\, u(n) = ca(e^{-aT})^n\, u(n)$

thus $H_d(e^{j\omega}) = \frac{ca}{1 - e^{-aT}\, e^{-j\omega}}$. For $H_d(e^{j0}) = 1$, $c = \frac{1 - e^{-aT}}{a}$

With this choice of c

$|H_d(e^{j\omega})|^2 = \frac{(1 - e^{-aT})^2}{1 - 2e^{-aT}\cos\omega + e^{-2aT}}$

thus $|H_d(e^{j\omega})|$ is:

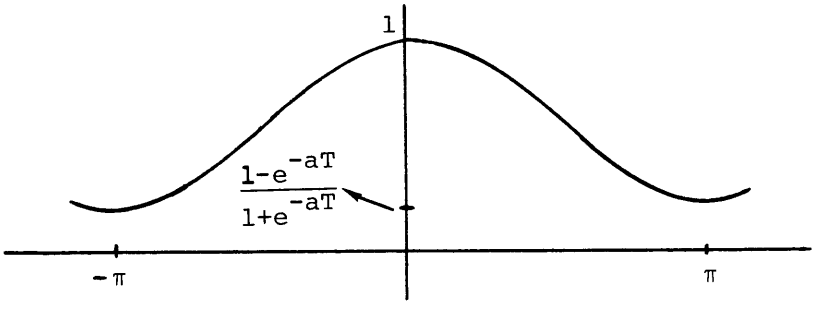


Figure S4.4-2

Note in particular that while the frequency response of the continuous-time filter asymptomatically approaches zero the frequency response of the digital filter doesn't. However as the sampling period T decreases, the value of $|H_d(e^{j\omega})|$ at $\omega = \pi$ decreases toward zero. The difference in the minimum values of $|H_a(j\Omega)|$ and $|H_d(e^{j\omega})|$ is of course due to aliasing.

Solution 4.5

From the discussion in the lecture, we know that

$$\tilde{X}_A(j\Omega) = \frac{1}{T}\sum_{r=-\infty}^{\infty} X_A\left(j\Omega + \frac{2\pi r}{T}\right)$$

and

$$X(e^{j\omega}) = \tilde{X}_A(j\Omega)\Big|_{\Omega T = \omega}$$

Let us assume that $X_A(j\Omega)$ has some arbitrary shape as indicated below. Since we are assuming that T is sufficiently small to prevent aliasing, $X_A(j\Omega)$ must be zero for $|\Omega| \geq \frac{\pi}{T}$. Then $\tilde{X}_A(j\Omega)$ and $X(e^{j\omega})$ are as shown in figure S4.5-1. $Y(e^{j\omega})$ corresponding to the output of the filter and $\tilde{Y}_A(j\Omega)$ and $Y_A(j\Omega)$ follow in a straightforward way and are as indicated in figure S4.5-1. Thus $y_A(n)$ could be obtained directly by passing x(n) through an ideal lowpass filter with unity gain in the passband and a cutoff frequency of $\frac{\pi}{4T}$ rad/sec. For the case in part (a) the cutoff frequency of the overall continuous-time filter is $\frac{\pi}{4} \times 10^4$ rad/sec and for the case in part (b) the cutoff frequency is $\frac{\pi}{8} \times 10^4$ rad/sec.

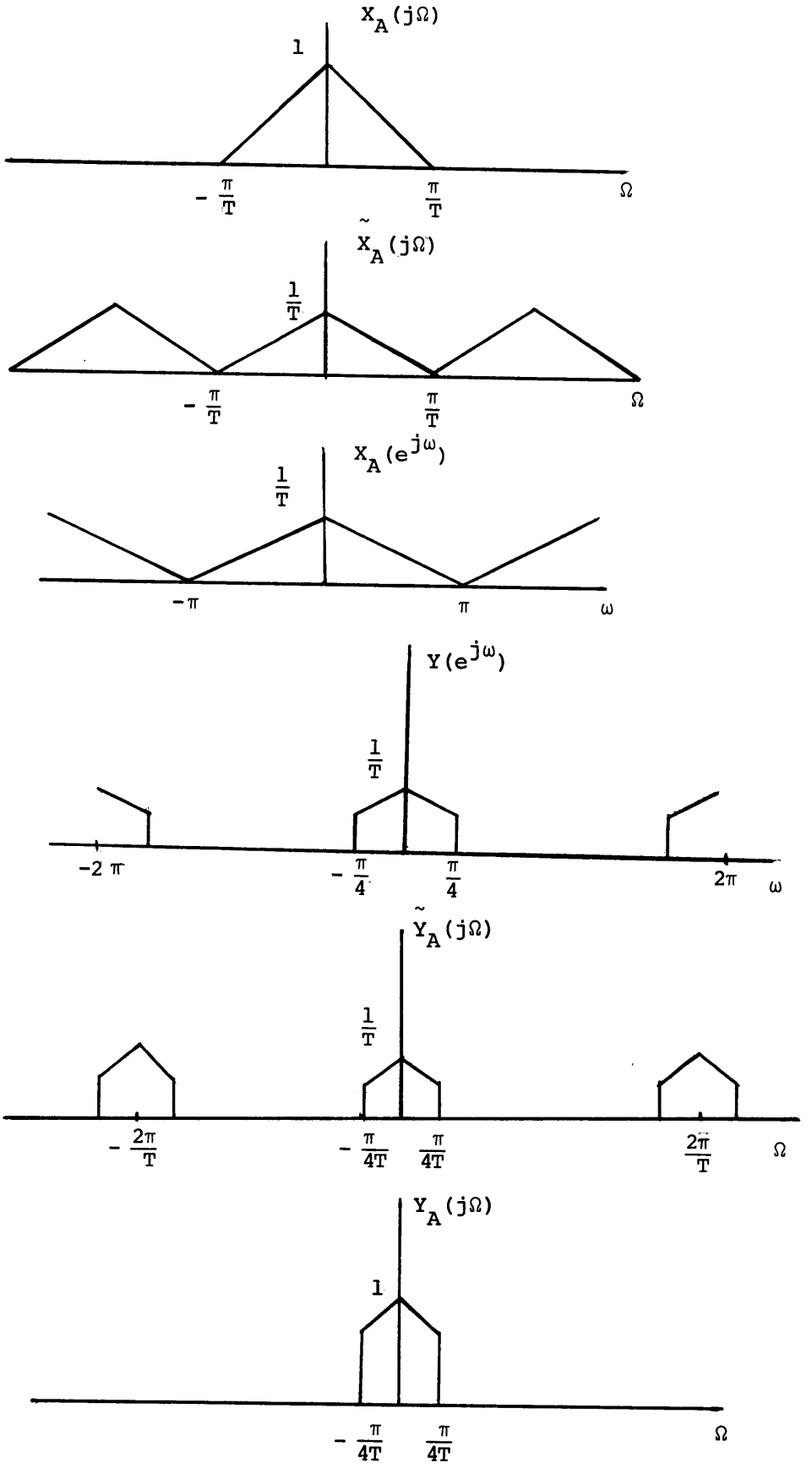

$X_A(j\Omega)$
1
$-\frac{\pi}{T}$
$\frac{\pi}{T}$
Ω
$\tilde{X}_A(j\Omega)$
$\frac{1}{T}$
$-\frac{\pi}{T}$
$\frac{\pi}{T}$
Ω
$X_A(e^{j\omega})$
$\frac{1}{T}$
$-\pi$
π
ω
$Y(e^{j\omega})$
$\frac{1}{T}$
-2π
$-\frac{\pi}{4}$
$\frac{\pi}{4}$
2π
ω
$\tilde{Y}_A(j\Omega)$
$\frac{1}{T}$
$-\frac{2\pi}{T}$
$-\frac{\pi}{4T}$
$\frac{\pi}{4T}$
$\frac{2\pi}{T}$
Ω
$Y_A(j\Omega)$
1
$-\frac{\pi}{4T}$
$\frac{\pi}{4T}$
Ω

Figure S4.5-1

Solution 4.6*

(1) and (2) can be verified by direct substitution into the inverse Fourier transform relation. (3) and (4) follow from (1) since

$\text{Re}\,[x(n)] = \frac{1}{2}\,[x(n) + x^*(n)]$ and $j\text{Im}\,[x(n)] = \frac{1}{2}\,[x(n) - x^*(n)]$.

(5) and (6) follow from (2) since $\text{Re}[X(e^{j\omega})] = \frac{1}{2}\,[X(e^{j\omega}) + X^*(e^{j\omega})]$

and $j\text{Im}\,[X(e^{j\omega})] = \frac{1}{2}\,[X(e^{j\omega}) - X^*(e^{-j\omega})]$.

Solution 4.7*

If $X(e^{j\omega})$ denotes the Fourier transform of $x(n)$, then

$$x(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, d\omega$$

Thus, with $y(n)$ denoting the convolution of $f(n)$ and $g(n)$ and since $Y(e^{j\omega}) = F(e^{j\omega})\, G(e^{j\omega})$, we wish to show that

$$y(0) = f(0)\, g(0) \quad .$$

But $$y(n) = \sum_{k=-\infty}^{+\infty} f(k)\, g(n - k)$$

so $$y(0) = \sum_{k=-\infty}^{+\infty} f(k)\, g(-k)$$

Since $f(k)$ is zero for $k < 0$ and $g(-k)$ is zero for $k > 0$,

$$\sum_{k=-\infty}^{+\infty} f(k)\, g(-k) = f(0)\, g(0)$$

Solution 4.8*

(a) Method A:

Consider $x(n)$ as a unit-sample $\delta(n)$. Then

$g(n)=h(n)$ and $r(n) = h(n) * g(-n) = \sum_{k=-\infty}^{+\infty} h(k)\, h(-n+k)$

Finally, $s(n) = r(-n) = \sum_{k=-\infty}^{+\infty} h(k)\, h(k + n)$

Consequently, $h_1(n) = \sum_{k=-\infty}^{+\infty} h(k)\, h(k + n)$

To show that this corresponds to zero phase, we wish to show that $h_1(n) = h_1(-n)$ since from Table 2.1 of the text with $h_1(n)$, if $h_1(n) = h_1(-n)$ then

$H_1(e^{j\omega}) = H_1^*(e^{j\omega})$ and hence the frequency response is real.

$$h_1(-n) = \sum_{k=-\infty}^{+\infty} h(k)\ h(k - n)$$

letting $k - n = r$,

$$h_1(-n) = \sum_{k=-\infty}^{+\infty} h(n + r)\ h(r)$$

which is identical to $h_1(n)$.

Alternatively we can show that $h_1(n)$ corresponds to a zero-phase filter by arguing in the frequency domain.
Specifically,

Let $\hat{g}(n) = g(-n)$. Then $\hat{G}_1(e^{j\omega}) = G_1^*(e^{j\omega}) = X^*(e^{j\omega})\ H^*(e^{j\omega})$

Also, $R(e^{j\omega}) = X^*(e^{j\omega})\ H\ (e^{j\omega})\ H^*(e^{j\omega}) = X^*(e^{j\omega})\ |H(e^{j\omega})|^2$

and $S(e^{j\omega}) = R^*(e^{j\omega})$

$= X(e^{j\omega})\ |H(e^{j\omega})|^2$

Thus, $H_1(e^{j\omega}) = |H(e^{j\omega})|^2$. Since $H_1(e^{j\omega})$ is real, it has a zero phase characteristic.

(b) Method B:

$G\ (e^{j\omega}) = X(e^{j\omega})\ H(e^{j\omega})$

$R(e^{j\omega}) = X^*(e^{j\omega})\ H(e^{j\omega})$

$Y(e^{j\omega}) = G\ (e^{j\omega}) + R^*(e^{j\omega})$

$= X(e^{j\omega})\ [H(e^{j\omega}) + H^*(e^{j\omega})]$

$= X(e^{j\omega})\ [2\ \text{Re}\ H(e^{j\omega})]$

Therefore $H_2(e^{j\omega}) = 2\ \text{Re}\ H(e^{j\omega}) = 2\ |H(e^{j\omega})|\ \cos[(\arg H(e^{j\omega})])$ and consequently is also zero phase.

(c) $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$ are sketched below. Clearly method A is the preferable method.

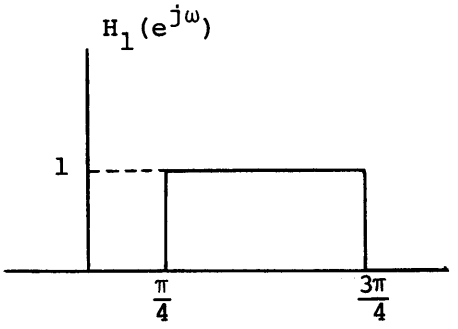


Figure S4.8-1

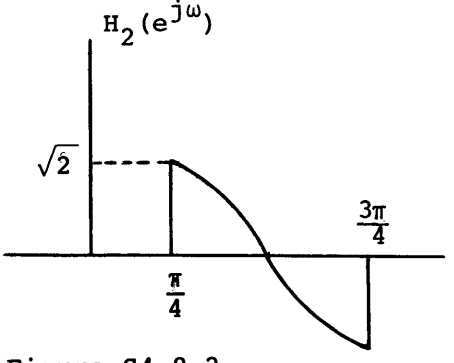


Figure S4.8-2

THE Z-TRANSFORM

1. Lecture 5 - 51 minutes

a.

$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$

$|X(e^{j\omega})| = \left|\sum_{-\infty}^{+\infty} x(n) e^{-j\omega n}\right|$

$\leq \sum_{-\infty}^{+\infty} |x(n)| \underbrace{|e^{-j\omega n}|}_{1}$

$X(e^{j\omega})$ converges if

$\sum_{-\infty}^{+\infty} |x(n)| < \infty$

stable system $\rightarrow$ $H(e^{j\omega})$ converges

Example

① $x(n) = (\frac{1}{2})^n u(n)$

$\sum_{-\infty}^{+\infty} |x(n)| = 2$

② $x(n) = (2)^n u(n)$

$\sum_{-\infty}^{+\infty} |x(n)| = \infty$

$X_r(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} [x(n) r^{-n}] e^{-j\omega n}$

$= \sum_{-\infty}^{+\infty} x(n) \underbrace{(re^{j\omega})^{-n}}_{z}$

The Z-Transform

$z = re^{j\omega}$

$X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n}$

$X(e^{j\omega}) = X(z)\big|_{z=e^{j\omega}}$

converges if

$\sum_{n=-\infty}^{+\infty} |x(n) r^{-n}| < \infty$

b.

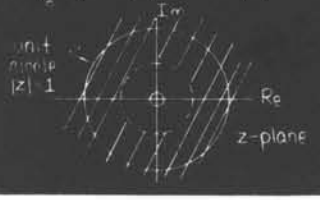
Example

$x(n) = (\frac{1}{2})^n u(n)$

$X(z) = \sum_{n=0}^{\infty} (\frac{1}{2})^n z^{-n} = \sum_{n=0}^{\infty} (\frac{1}{2} z^{-1})^n$

$= \frac{1}{1 - \frac{1}{2} z^{-1}} = \frac{z}{z - \frac{1}{2}}$

$\sum_{0}^{\infty} |(\frac{1}{2} z^{-1})^n| < \infty \Rightarrow |z| > \frac{1}{2}$



Example

$x(n) = -(\frac{1}{2})^n u(-n-1)$

$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} = \frac{z}{z - \frac{1}{2}}$

$\sum_{-1}^{-\infty} |(\frac{1}{2} z^{-1})^n| < \infty \Rightarrow |z| < \frac{1}{2}$

(1) Region of Convergence bounded by poles or (0/∞)

(2) Finite Length Sequences

$0 < |z| < \infty$

(3) Right-sided $x(n) = 0$ $n < n_1$

$R_{x-} < |z| < \infty$

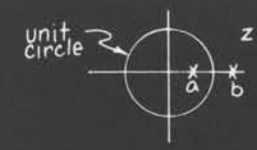
(4) Left-sided $x(n) = 0$ $n > n_1$

$0 < |z| < R_{x+}$

(5) Two-sided

$R_{x-} < |z| < R_{x+}$

c.



Region of Convergence	Which Sided	Fourier Transform?
$\|z\| < a$	left	no
$a < \|z\| < b$	two	yes
$b < \|z\|$	right	no

$y(n) = x(n) * h(n)$

$Y(z) = X(z)\ H(z)$

$H(z) \triangleq$ System Function

✓Stable $\longleftrightarrow$ unit circle in R. of C.

causal $\Longrightarrow$ h(n) right-sided

$\Longrightarrow$ R. of C. outside outermost pole

Example

$y(n) - \frac{1}{2} y(n-1) = x(n)$

useful property:

$y(n) \longleftrightarrow Y(z)$

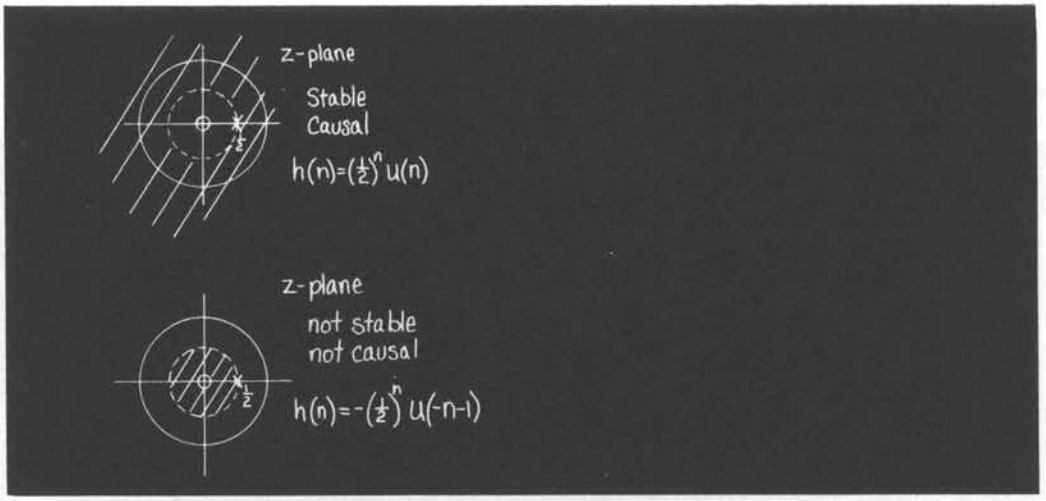
$y(n+n_0) \longleftrightarrow z^{n_0} Y(z)$

$Y(z) - \frac{1}{2} z^{-1} Y(z) = X(z)$

$H(z) = Y(z)/X(z)$

$= \frac{1}{1 - \frac{1}{2} z^{-1}}$

d.



2. Comments

The Fourier transform does not converge for all sequences. Specifically, depending on the definition of convergence used, the sequence is required to be either absolutely summable or have finite energy. It is useful to note that for an LSI system the condition for stability corresponds to the condition for convergence of the Fourier transform of the unit-sample response (i.e. the frequency response). The z-transform represents a generalization of the Fourier transform to include sequences for which the Fourier transform doesn't converge; and can be interpreted as the Fourier transform of the sequence modified by multiplication with a complex exponential.

Associated with the z-transform is a region of convergence, i.e. a set of values of z for which the transform converges. As illustrated in the lecture two different sequences can have z-transforms of the same functional form and thus for which the z-transforms are distinguished only by their regions of convergence. This emphasizes the importance of the region of convergence in specifying the z-transform. Based on the properties of the region of convergence it is often possible to indicate implicitly the region of convergence for the z-transform. For example an indication that a sequence is right-sided implies that the region of convergence lies outside the outermost pole in the z-plane. Corresponding statements can be made about left-sided and two-sided sequences. Also, for a "stable" sequence, the region of convergence includes the unit circle in the z-plane. Thus if it is indicated that a z-transform corresponds to a stable sequence then its region of convergence is implied.

3. Reading

Text: Read sections 4.0 (page 149) through 4.2.

4. Problems

Problem 5.1

Determine which of the following sequences have Fourier transforms which converge:

(i) $x(n) = 2^n u(n)$

(ii) $x(n) = 2^n u(-n)$

(iii) $x(n) = (\frac{1}{2})^n u(n)$

(iv) $x(n) = u(n)$

Problem 5.2

(a) For each of the following sequences sketch the pole-zero pattern of their z-transforms. Include an indication of the region of convergence.

(i) $\delta(n) + (\frac{1}{2})^n u(n)$

(ii) $(\frac{1}{3})^{-n} u(n)$

(iii) $(\frac{1}{2})^n u(n) + (\frac{1}{3})^n u(n)$

(iv) $\delta(n) - \frac{1}{8} \delta(n - 3)$

(v) $(\frac{1}{2})^{|n|}$

(b) From your results in part (a) determine which of those sequences could correspond to the unit sample response of a stable system.

Problem 5.3

Consider a sequence x(n) with z-transform X(z). The pole zero pattern for X(z) is shown below:

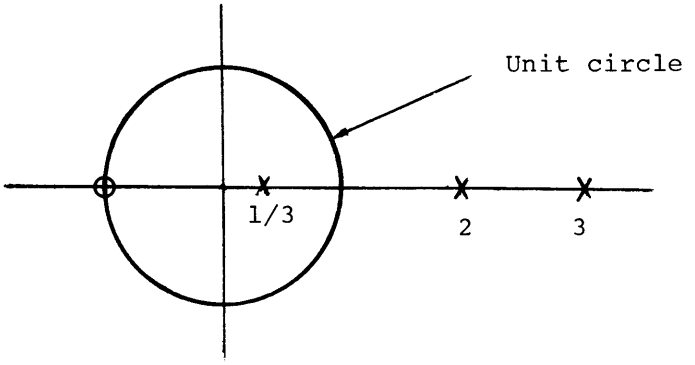


Figure P5.3-1

(a) Determine the region of convergence of X(z) if it is known that the Fourier transform of the sequence converges. For this case determine also whether the sequence is right-sided, left-sided or two-sided.

(b) If it is not known whether the Fourier transform of x(n) converges but the sequence is known to be two-sided how many different possible sequences could the pole-zero pattern of figure P5.3-1 correspond to? For each of these possibilities indicate the associated region of convergence.

Problem 5.4

Let X(z) denote the z-transform of x(n). Determine in terms of X(z) the z-transform of $x(n + n_0)$. If you have difficulty deciding how to proceed you may find it helpful to first consider an example such as $x(n) = (\frac{1}{2})^n u(n)$ and $n_0 = -1$.

Problem 5.5*

Determine the z-transform of each of the following. Include with your answer the region of convergence in the z-plane and a sketch of the pole-zero pattern. Express all sums in closed form. α can be complex.

(a) $x(n) = \alpha^{|n|}$, $0 < |\alpha| < 1$

(b) $x(n) = Ar^n \cos(\omega_0 n + \phi)\, u(n)$, $0 < r < 1$.

(c) $$x(n) = \begin{cases} 1, & 0 \le n < N - 1, \\ 0, & N \le n, \\ 0, & n < 0. \end{cases}$$

(d) $$x(n) = \begin{cases} n, & 0 \le n \le N, \\ 2N - n, & N + 1 \le n \le 2N, \\ 0, & 2N \le n, \\ 0, & 0 > n. \end{cases}$$

[Hint (easy way): First express x(n) in terms of the x(n) in part (c).]

Problem 5.6*

For a sequence x(n) which is zero for $n < 0$, show that $\lim_{z \to \infty} X(z) = x(0)$. What is the corresponding theorem if the sequence is zero for $n > 0$?

Problem 5.7*

Let x(n) denote a causal sequence; i.e., $x(n) = 0$, $n < 0$. Furthermore, assume that $x(0) \neq 0$.

(a) Show that there are no poles or zeros of X(z) at $z = \infty$.

(b) Show that the number of poles in the finite z-plane equals the number of zeros in the finite z-plane. (The finite z-plane excludes $z = \infty$.)

THE Z-TRANSFORM

Solution 5.1

For convergence of the Fourier transform, the sequence must be absolutely summable or square summable, (i.e. have finite energy) depending on the type of convergence to be considered.

In this problem, sequences (i) and (iv) are neither absolutely summable nor square summable, and thus their Fourier transforms do not converge. Sequences (ii) and (iii) are both absolutely summable and square summable.

Solution 5.2

(a)

(i) $$X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} \delta(n) z^{-n} + \sum_{n=-\infty}^{+\infty} \left(\frac{1}{2}\right)^n u(n) z^{-n}$$

$$= 1 + \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n = \frac{1}{1 - \frac{1}{2} z^{-1}} \quad \text{for } \left|\frac{1}{2} z^{-1}\right| < 1 .$$

$$\therefore \quad X(z) = 1 + \frac{1}{1 - \frac{1}{2} z^{-1}} = \frac{2 - \frac{1}{2} z^{-1}}{1 - \frac{1}{2} z^{-1}}; \quad |z| > \frac{1}{2}$$

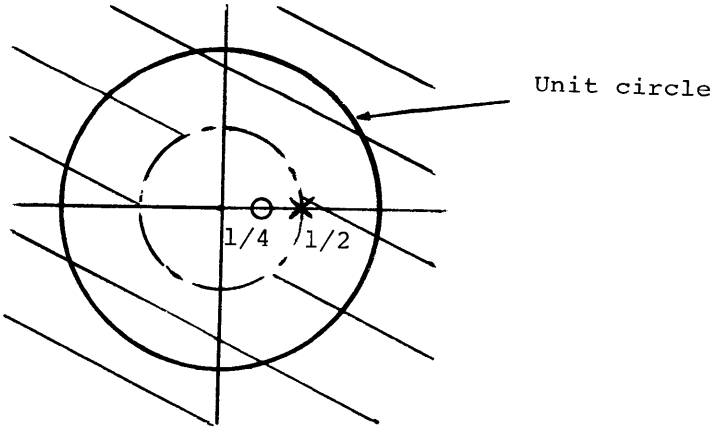


Figure S5.2-1

We obtain the z-transforms of the remaining sequences in a similar manner:

(ii) $X(z) = \dfrac{1}{1 - 3z^{-1}}; \quad |z| > 3$

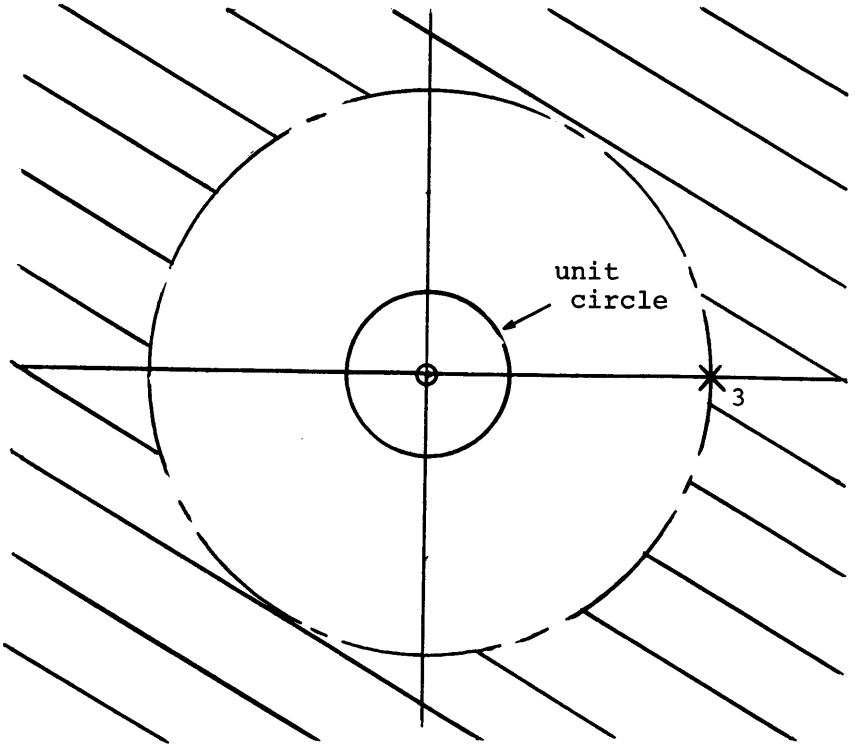


Figure S5.2-2

(iii) $X(z) = \dfrac{1}{1 - \frac{1}{2}z^{-1}} + \dfrac{1}{1 - \frac{1}{3}z^{-1}}; \qquad |z| > \frac{1}{2}$

$$= \frac{(2 - \frac{5}{6}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})}; \qquad |z| > \frac{1}{2}$$

(iv) $X(z) = 1 - \frac{1}{8}z^{-3}; \; z \neq 0$

There is a third order pole at z = 0. The zeros occur at the three cube roots of (1/8), i.e. at

$$z = \frac{1}{2}, \; \frac{1}{2}\, e^{j\frac{2\pi}{3}}, \; \frac{1}{2}\, e^{-j\frac{2\pi}{3}}$$

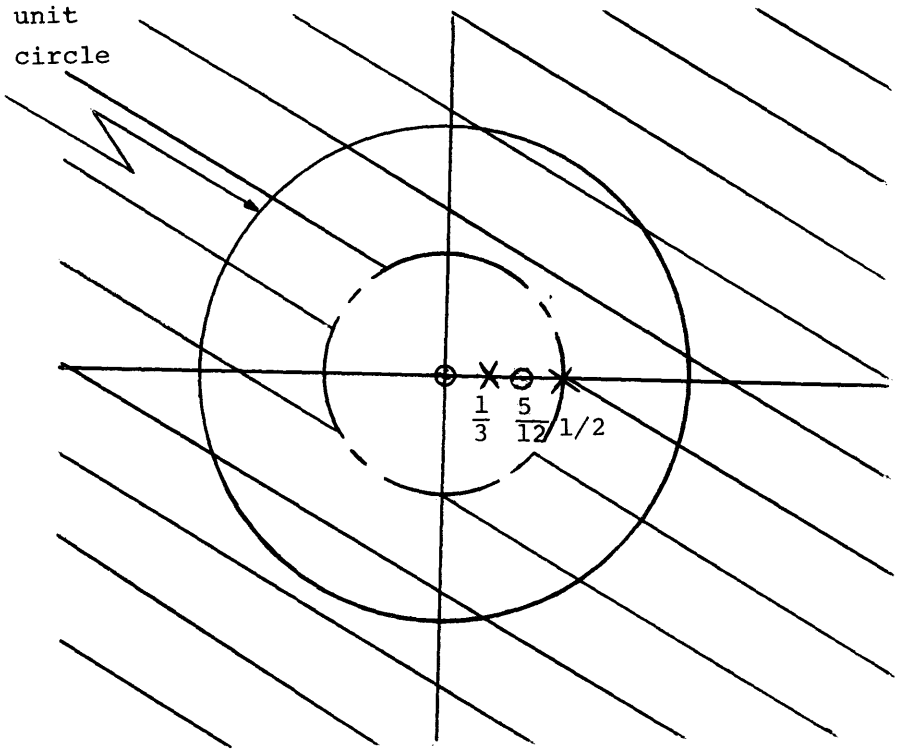
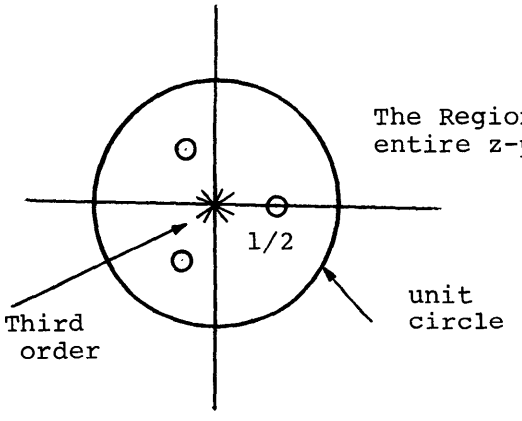


Figure S5.2-3



The Region of convergence is entire z-plane except z = 0

unit
circle

Figure S5.2-4

$$(v) \quad X(z) = \sum_{-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} z^{-n} + \sum_{0}^{\infty} \left(\frac{1}{2}\right)^{n} z^{-n}$$

$$= \sum_{0}^{\infty} \left(\frac{1}{2}\right)^{n} z^{n} + \sum_{0}^{\infty} \left(\frac{1}{2}\right)^{n} z^{-n} - 1$$

$$\sum_{0}^{\infty} \left(\frac{1}{2}\right)^{n} z^{n} = \frac{1}{1 - \frac{1}{2}z} \; ; \quad |z| < 2$$

$$\sum_{0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \frac{1}{1 - \frac{1}{2}z^{-1}}; \quad |z| > \frac{1}{2}$$

$$\therefore X(z) = \frac{1}{1 - \frac{1}{2}z} + \frac{1}{1 - \frac{1}{2}z^{-1}} - 1; \quad \frac{1}{2} < |z| < 2$$

$$= \frac{-\frac{3}{2}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})}; \quad \frac{1}{2} < |z| < 2$$

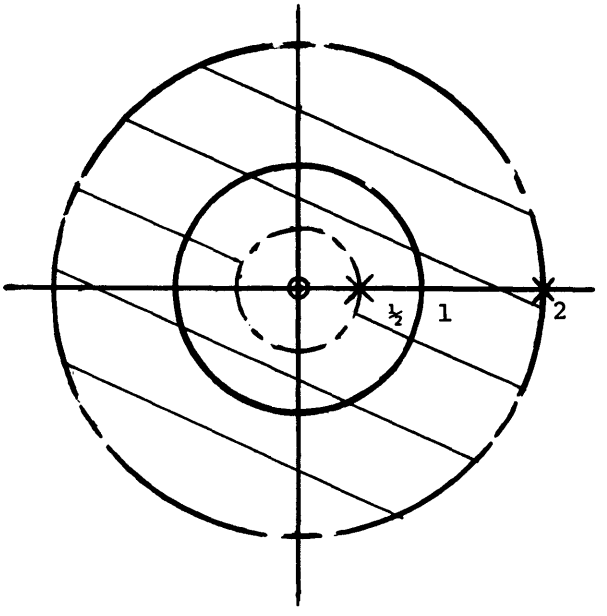


Figure S5.2-5

(b) In order that a sequence correspond to the unit sample response of a stable system the region of convergence must include the unit circle. Thus only sequence (ii) would not correspond to a stable system.

Solution 5.3

(a) If the Fourier transform converges then the region of convergence includes the unit circle. It is bounded by two circles extending outward and inward to the nearest poles. Thus the region of convergence is $\frac{1}{3} < |z| < 2$. Since it does not extend to zero or to infinity, the sequence is two-sided.

(b) If the sequence is two-sided then the region of convergence must be bounded by poles. Thus, the given pole-zero pattern with the region of convergence either as $\frac{1}{3} < |z| < 2$ or as $2 < |z| < 3$ will correspond to two-sided sequences.

Solution 5.4

Let $X_0(z)$ denote the z-transform of $x(n + n_0)$

then $X_0(z) = \sum_{n=-\infty}^{+\infty} x(n + n_0) z^{-n}$

let $m = n + n_0$. Then

$$X_0(z) = \sum_{m=-\infty}^{+\infty} x(m) z^{n_0} z^{-m}$$

$$= z^{n_0} \sum_{m=-\infty}^{+\infty} x(m) z^{-m}$$

$$= z^{n_0} X(z)$$

Solution 5.5*

(a) $$X(z) = \frac{z(1-\alpha^2)}{(1-\alpha z)(z-\alpha)} \; ; \; |\alpha| < |z| < \frac{1}{|\alpha|}$$

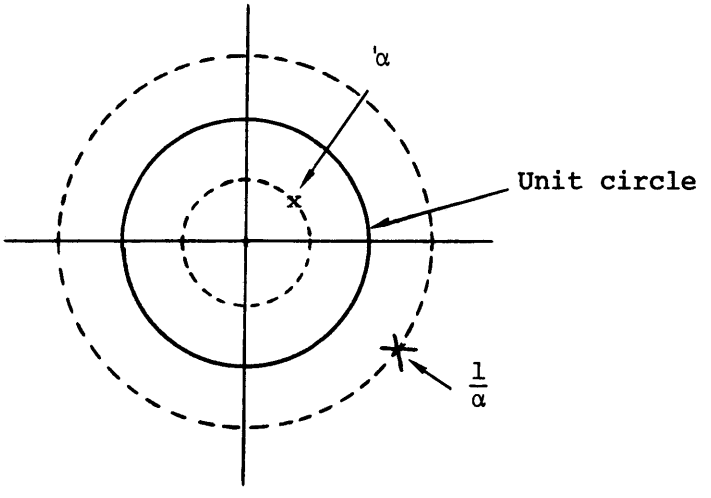


Figure S5.5-1

(b) $$X(z) = \frac{Az}{2} \left[\frac{2z \cos \phi - 2r \cos(\omega_0 - \phi)}{(z - r e^{j\omega_0})(z - r e^{-j\omega_0})} \right]; \quad |z| > r$$

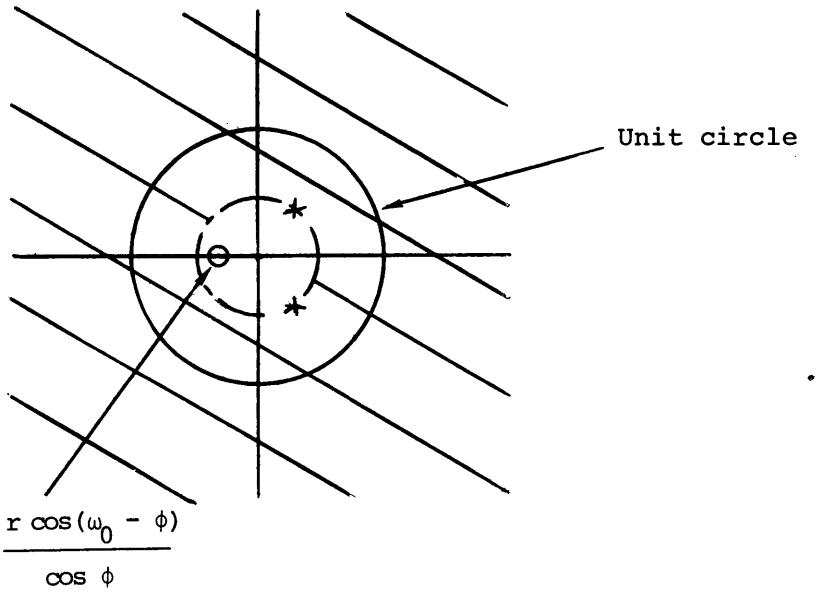


Figure S5.5-2

(c) $X(z) = \dfrac{z^N - 1}{z^{N-1}(z-1)} \qquad z \neq 0$

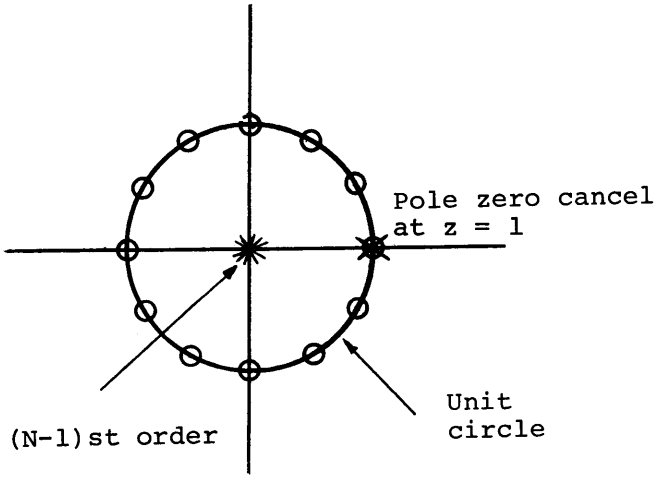


Figure S5.5-3

(d) $x_d(n+1) = x_c(n) * x_c(n)$ where $x_d(n)$ and $x_c(n)$ are the sequences in part (d) and (c) respectively. Therefore

$$X_d(z) = z^{-1} X_c(z) X_c(z)$$

Therefore the zeros in the pole-zero plot of part (c) become double zeros and there is now a (2N - 1)st order pole at z = 0.

Solution 5.6*

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$
$$= X(0) + x(1) z^{-1} + \ldots .$$

For $z \to \infty$ all of the terms in this sum approach zero with the exception of the first term. Thus

$$\lim_{z \to \infty} X(z) = x(0) \quad \text{for } x(n) \text{ causal.}$$

If $x(n) = 0$ $n > 0$ then the corresponding relation is

$$\lim_{z \to 0} X(z) = x(0) \quad .$$

Solution 5.7*

(a) From problem 5.7 we know that $\lim_{z \to \infty} X(z)$ is $x(0)$. Assuming that $x(0)$ is finite, then, $\lim_{z \to \infty} X(z)$ is finite and consequently there are no poles of $X(z)$ at $z = \infty$. Furthermore, since $x(0) \neq 0$ there are no zeros of $X(z)$ at $z = \infty$.

(b) Since $x(n) = 0$; $n < 0$ $X(z)$ is a rational function of z^{-1}, i.e. it is of the form

$$X(z) = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{n=0}^{N} a_n z^{-n}}$$

The numerator factor introduces M zeros and also an $M^{\underline{th}}$ order pole at $z = 0$. The denominator factor introduces N poles and an $N^{\underline{th}}$ order zero at $z = 0$. Thus there are a total of $(M + N)$ zeros and $(M + N)$ poles i.e. the total number of poles and zeros in the entire z-plane (including $z = \infty$) is the same.

From part (a) of this problem we know that there are no poles or zeros at $z = \infty$. Thus the number of poles and zeros in the finite z-plane is the same.

THE INVERSE Z-TRANSFORM

1. Lecture 6 - 47 minutes

a.

$$X(z)=\sum_{n=-\infty}^{+\infty} x(n)z^{-n}$$

Inverse z-transform

Inspection Method

$a^n u(n) \longleftrightarrow \frac{1}{1-az^{-1}} \quad |z|>|a|$

$-a^n u(-n-1) \longleftrightarrow \frac{1}{1-az^{-1}} \quad |z|<|a|$

ETC.

2. Power Series

$$X(z)=\frac{1}{1-az^{-1}}=\frac{z}{z-a}$$

$|z|>|a|$

$1-az^{-1}\,\overline{)\,1}\;\;1+az^{-1}+a^2z^{-2}+a^3z^{-3}+\cdots$

$\frac{1-az^{-1}}{az^{-1}}$

$\frac{az^{-1}-a^2z^{-2}}{a^2z^{-2}}$

$|z|<|a|$

but also

$$\frac{1}{1-az^{-1}}=-a^{-1}z-a^{-2}z^2+\cdots$$

3. Partial Fraction Exp.

$$F(x)=\frac{P(x)}{Q(x)}=\sum_{k=1}^{N}\frac{R_k}{x-x_k}$$

$$F(x)(x-x_r)\Big|_{(x=x_r)}$$

$$=\sum_{k=1}^{N}\underbrace{\frac{R_k}{(x-x_k)}(x-x_r)\Big|_{(x=x_r)}}_{\begin{array}{l}=0\quad k\neq r\\ =R_r\quad k=r\end{array}}$$

$$R_r=F(x)(x-x_r)\Big|_{(x=x_r)}$$

= Residue of $F(x)$ at x_r

b.

$X=z \quad X(z)=\sum_k \frac{A_k}{z-a_k}$

$X=z^{-1} \quad X(z)=\sum_k \frac{B_k}{1-a_k z^{-1}}$

Example

$$X(z)=\frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} \quad |z|>\tfrac{1}{2}$$

let $X \Rightarrow z^{-1}$

$$\frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}=\frac{8}{(z^{-1}-2)(z^{-1}-4)}$$

$$\frac{8}{(z^{-1}-2)(z^{-1}-4)}(z^{-1}-2)\Big|_{z^{-1}=2}=-4$$

$$\frac{8}{(z^{-1}-2)(z^{-1}-4)}(z^{-1}-4)\Big|_{z^{-1}=4}=4$$

$$X(z)=\frac{-4}{z^{-1}-2}+\frac{4}{z^{-1}-4}$$

$$=\underbrace{\frac{2}{1-\frac{1}{2}z^{-1}}}_{2(\frac{1}{2})^n u(n)}+\underbrace{\frac{-1}{1-\frac{1}{4}z^{-1}}}_{-(\frac{1}{4})^n u(n)}$$

4. Contour Integration

$$x(n)=\frac{1}{2\pi j}\oint_c X(z)\,z^{n-1}\,dz$$

$=\sum$ (residues of $X(z)z^{n-1}$ at poles inside c)

Example

$$X(z)=\frac{1}{1-\frac{1}{2}z^{-1}}$$

$$=\frac{z}{z-\frac{1}{2}} \quad |z|>\tfrac{1}{2}$$

c.

unit circle

z-plane

$X(z)z^{n-1}=z^n/(z-\frac{1}{2})$

$n\geq 0$ 1 pole at $z=\frac{1}{2}$

Residue of $\frac{z^n}{z-\frac{1}{2}}$ at $z=\frac{1}{2}$

$$\frac{z^n}{(z-\frac{1}{2})}(z-\tfrac{1}{2})\Big|_{z=\frac{1}{2}}=(\tfrac{1}{2})^n$$

$n\geq 0 \quad x(n)=(\frac{1}{2})^n$

$n<0$ 1 pole at $z=\frac{1}{2}$

n poles at $z=0$

Easy way:

$$x(n)=\frac{1}{2\pi j}\oint_c X(z)z^{n-1}dz$$

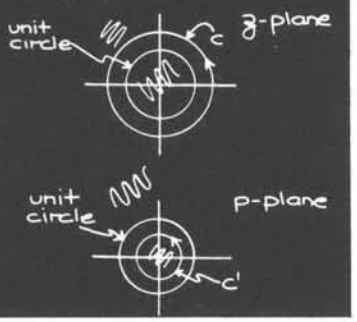
let $z=p^{-1} \quad z^{n-1}=p^{-n+1}$

If $z=re^{j\theta}$

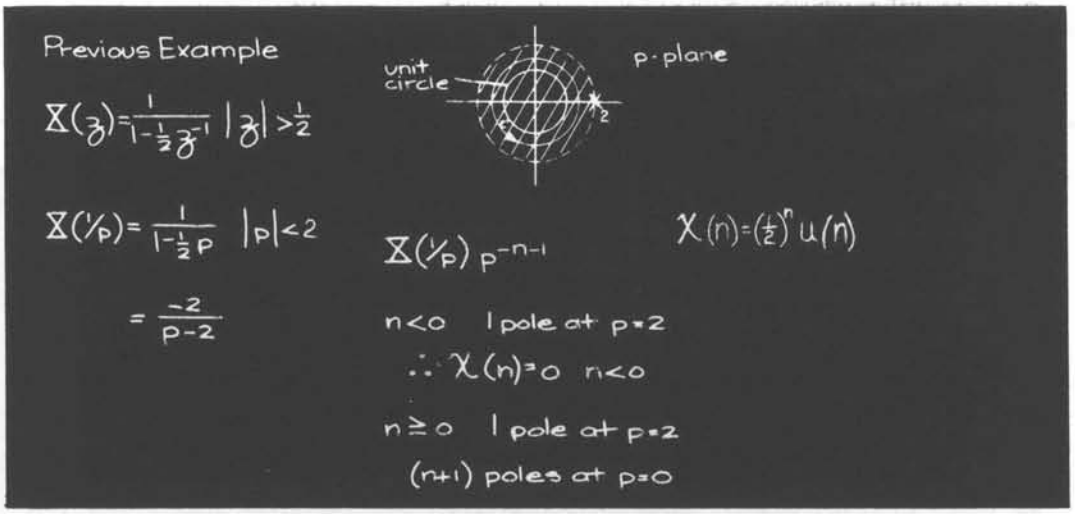
then $p=(1/r)e^{-j\theta}$

$$x(n)=\frac{-1}{2\pi j}\oint_{c'} X(1/p)p^{-n-1}dp$$

$$=\frac{1}{2\pi j}\oint_{c'} X(1/p)p^{-n-1}dp$$



d.



Correction (Fig. d.): The region of convergence of X(1/p) is $|p| < 2$ not $|p| < 1/2$.

2. Comments

This lecture presents several techniques for obtaining a sequence from its z-transform. The primary focus is on informal methods. The first of these, referred to as the inspection method corresponds to utilizing the fact that simple z-transforms and the sequences that generate them are recognizable by inspection. An extension of this method consists of expanding a more complicated z-transform in a partial fraction expansion and then recognizing the sequences that correspond to the individual terms.

A somewhat different method for obtaining the inverse z-transform consists of expanding the z-transform as a power series, utilizing either positive or negative values of z, as dictated by the region of convergence and recognizing the coefficients in the series expansion as corresponding to the sequence values. One of the drawbacks to this method is that except in simple cases it does not lead to a closed form expression for the sequence. The major advantage with it is that it is more easily applied to z-transform expressions that are non-rational functions of z than are other methods.

The final method presented in this lecture is the use of the formal inverse z-transform relationship consisting of a contour integral in the z-plane. This contour integral expression is derived in the text and is useful, in part, for developing z-transform properties and theorems. The mechanics of evaluating the inverse z-transform rely on the use

of residue calculus. An important point stressed in the lecture is the fact that the inverse z-transform integral is valid for both positive and negative values of n. However, for n negative there are multiple order poles introduced at the origin, the evaluation of the residues for which is cumbersome. In the lecture a procedure for avoiding this through a substitution of variables is introduced.

3. Reading

Text: Sections 4.3 (page 165) and 4.5.

4. Problems

Problem 6.1

Listed below are several z-transforms. Determine by "inspection" the sequences to which they correspond.

(i) $X(z) = 1;$ $\quad |z| \leq \infty$

(ii) $X(z) = z^3;$ $\quad |z| < \infty$

(iii) $X(z) = \dfrac{1}{1-az^{-1}};$ $\quad |z| > |a|$

(iv) $X(z) = \dfrac{1}{1-az^{-1}};$ $\quad |z| < |a|$

(v) $X(z) = -2z^{-2} + 1 + 2z;$ $\quad |z| < +\infty$

Problem 6.2

Listed below are three z-transforms. For each determine the inverse z-transform by using contour integration, and again by using a partial fraction expansion.

(i) $X(z) = \dfrac{1}{1+\frac{1}{2}z^{-1}};$ $\quad |z| > \frac{1}{2}$

(ii) $X(z) = (1 - \frac{1}{2}z^{-1}) \Big/ \left[1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right];$ $\quad |z| > \frac{1}{2}$

(iii) $X(z) = \dfrac{1-2z^{-1}}{z^{-1}-2};$ $\quad |z| > \frac{1}{2}$

Problem 6.3

By using a power series expansion determine a sequence x(n) whose z-transform is

$X(z) = e^z$

Problem 6.4*

Given here are four z-transforms. Determine which ones could be the transfer function of a discrete linear system which is not necessarily stable but for which the unit-sample response is zero for $n < 0$. Clearly state your reasons.

(a) $(1 - z^{-1})^2 / (1 - \frac{1}{2}z^{-1})$.

(b) $(z - 1)^2 / (z - \frac{1}{2})$.

(c) $(z - \frac{1}{4})^5 / (z - \frac{1}{2})^6$.

(d) $(z - \frac{1}{4})^6 / (z - \frac{1}{2})^5$.

Problem 6.5*

In Sec. 4.5 of the text we discussed the fact that for $n < 0$ it is often more convenient to evaluate the inverse transform relation of Eq. (4.67) of the text by using a substitution of variables $z=p^{-1}$ to obtain Eq. (4.74). If the contour of integration is the unit circle, for example, this has the effect of mapping the outside of the unit circle to the inside, and vice versa. It is worth convincing yourself, however, that without this substitution, we can obtain $x(n)$ for $n < 0$ by evaluating the residues at the multiple-order poles at $z = 0$.

Let

$$X(z) = \frac{1}{1 - \frac{1}{2}z}$$

where the region of convergence includes the unit circle.

(a) Determine $x(0)$, $x(-1)$, and $x(-2)$ by evaluating Eq. (4.67) explicitly, obtaining the residue for the poles at $z = 0$.

(b) Determine $x(n)$ for $n > 0$ by evaluating expression (4.67) and for $n \leq 0$ by evaluating Eq. (4.74).

THE INVERSE Z-TRANSFORM

Solution 6.1

(i) $x(n) = \delta(n)$

(ii) $x(n) = \delta(n + 3)$

(iii) $x(n) = a^n\, u(n)$

(iv) $x(n) = -a^n\, u(-n-1)$

(v) $x(n) = -2\,\delta(n - 2) + \delta(n) + 2\,\delta(n + 1)$

Solution 6.2

(i) Using contour integration,

$$x(n) = \frac{1}{2\pi j}\oint_C \frac{1}{1 + \frac{1}{2}z^{-1}}\, z^{n-1}\, dz$$

$$= \frac{1}{2\pi j}\oint_C \frac{z^n}{z + \frac{1}{2}}\, dz$$

Since the contour of integration must lie inside the region of convergence, i.e. for $|z| > \frac{1}{2}$, it encloses the pole at $z = -\frac{1}{2}$. For $n \geq 0$ there are no poles at $z = 0$. Thus, for $n \geq 0$

$$x(n) = \text{Residue of } \frac{z^n}{z + \frac{1}{2}} \quad \text{at} \quad z = -\frac{1}{2} \quad \text{or}$$

$$x(n) = (-\tfrac{1}{2})^n \quad n \geq 0$$

For $n < 0$ we use the substitution of variables. Thus

$$x(n) = \frac{1}{2\pi j}\oint_{C'} X(1/p)\, p^{-n-1}\, dp$$

where now the contour of integration must lie inside the region of convergence of $X(1/p)$ i.e. for $|p| < 2$. Then

$$x(n) = \frac{1}{2\pi j}\oint_{C'} \frac{2p^{-n-1}}{p + 2}\, dp$$

For n negative the only pole is at $p = -2$ which is outside the contour of integration. Thus for $n < 0$ $x(n) = 0$. Combining these two results, then

$$x(n) = (-\tfrac{1}{2})^n\, u(n)$$

Since for this example $X(z)$ has only a single pole, the partial fractions expansion method wouldn't apply. The inspection method would and in fact corresponds to problem 6.1 (iii) for $a = -\frac{1}{2}$.

(ii) Using contour integration,

$$x(n) = \frac{1}{2\pi j}\oint_c \left[z(z - \tfrac{1}{2}) \Big/ (z + \tfrac{1}{2})(z + \tfrac{1}{4}) \right] z^{n-1}\, dz$$

where on the contour c, $|z| > \frac{1}{2}$.

For $n \geq 0$

$$x(n) = \left(\text{Residue of } X(z)z^{n-1} \text{ at } z = -\tfrac{1}{2}\right) + \left(\text{Residue of } X(z)z^{n-1} \text{ at } z = -\tfrac{1}{4}\right)$$

$$= 4(-\tfrac{1}{2})^n - 3(-\tfrac{1}{4})^n$$

For $n < 0$

$$x(n) = \frac{1}{2\pi j}\oint \frac{1-\frac{1}{2}p}{(1+\frac{1}{4}p)(1+\frac{1}{2}p)}\, p^{-n-1}\, dp$$

The contour of integration is $|p| < 2$. The poles of the integrand for $n < 0$ are at $p = -4$ and $p = -2$. Therefore there are no poles inside the contour of integration and thus

$x(n) = 0$ $n < 0$, i.e.

$$x(n) = \left[4(-\tfrac{1}{2})^n - 3(-\tfrac{1}{4})^n\right] u(n)$$

Using a partial fractions expansion,

$$X(z) = \frac{1-\frac{1}{2}z^{-1}}{(1+\frac{1}{4}z^{-1})(1+\frac{1}{2}z^{-1})}$$

$$= \frac{8 - 4z^{-1}}{(z^{-1} + 4)(z^{-1} + 2)} = \frac{a}{z^{-1} + 4} + \frac{b}{z^{-1} + 2}$$

$$a = X(z)(z^{-1} + 4)\Big|_{z^{-1} = -4} = -12$$

$$b = X(z)(z^{-1} + 2)\Big|_{z^{-1} = -2} = 8$$

So, $$X(z) = \frac{-12}{z^{-1} + 4} + \frac{8}{z^{-1} + 2} = \frac{-3}{(1+\frac{1}{4}z^{-1})} + \frac{4}{(1+\frac{1}{2}z^{-1})}$$

With X(z) expressed in this form, x(n) can be obtained by inspection. Specifically, since the region of convergence of X(z) is $|z| > \frac{1}{2}$ x(n) is a right-sided sequence and is given by

$$x(n) = \left[-3(-\tfrac{1}{4})^n u(n) + 4(-\tfrac{1}{2})^n\right] u(n) \quad .$$

(iii) Using contour integration,

$$x(n) = \frac{1}{2\pi j} \oint_{C'} \frac{z - 2}{1 - 2z} z^{n-1} \, dz$$

for $n > 0$ there is one pole, at $z = \frac{1}{2}$. Thus for $n > 0$

$$x(n) = -\frac{1}{2}(z - 2)z^{n-1} \Big|_{z = \frac{1}{2}} = +\frac{3}{2} \left(\frac{1}{2}\right)^n$$

For $n = 0$ there is also a pole at $z = 0$. Hence for $n = 0$

$$\text{residue of } x(n) \text{ at the origin} = \frac{z - 2}{1 - 2z} \Big|_{z = 0} = -2$$

For $n < 0$ following the same procedure as in (i) and (ii), $x(n) = 0$. Thus

$$x(n) = \frac{3}{2} \left(\frac{1}{2}\right)^n u(n) - 2\delta(n)$$

Using the partial fractions expansion, we note first that the order of the numerator and denominator are equal, and thus the partial fractions expansion of $X(z)$ is in the form

$$X(z) = a + \frac{b}{z^{-1} - 2}$$

To obtain the constant a we use long division. Thus:

$$\begin{array}{r} -2 \\ z^{-1} - 2 \overline{)\,-2z^{-1} + 1} \\ \underline{-2z^{-1} + 4} \\ -3 \end{array}$$

Therefore $X(z) = -2 + \frac{-3}{z^{-1} - 2} = -2 + \frac{3/2}{1 - \frac{1}{2}z^{-1}}$

Since the region of convergence is $|z| > \frac{1}{2}$, the inverse z-transform is (by inspection)

$$x(n) = -2\delta(n) + \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)^n u(n)$$

Solution 6.3

Expanding e^z in a power series, we obtain

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^{-n}$$

Thus, $x(n) = \frac{1}{(-n)!} u(-n)$

Solution 6.4

If $h(n) = 0$ for $n < 0$, then $H(z)$ must be expressible as a power series of the form

$$H(z) = \sum_{n=0}^{\infty} h(n)\, z^{-n} \quad .$$

Specifically, it cannot contain any positive powers of z. Consequently, expressed as a ratio of polynomials in z, the order of the numerator must be less than or equal to the order of the denominator. Equivalently, we can refer to the result of problem 5.

In that problem we showed that if $x(n) = 0$ $n < 0$ then $\lim_{z\to\infty} X(z)$ must be finite. Applying either of these conditions, (a) and (c) could correspond to causal systems while (b) and (d) could not.

Solution 6.5

(a)

$$x(n) = \frac{1}{2\pi j} \oint_c \frac{1}{1 - \frac{1}{2}z}\, z^{n-1}\, dz$$

Since the region of convergence includes the unit circle and the pole is at $z = 2$, $|z| < 2$ on the contour c . Therefore the pole at $z = 2$ is not inside the contour of integration. Hence for $n \leq 0$ the only poles inside the contour of integration are at $z = 0$:

$$x(0) = \text{Res}\left[\frac{1}{z(1 - \frac{1}{2}z)} \quad \text{at } z = 0\right] = 1$$

$$x(-1) = \text{Res}\left[\frac{1}{z^2(1 - \frac{1}{2}z)} \quad \text{at } z = 0\right]. \quad \text{Applying eq. (4.44) of the text.}$$

we obtain

$$x(-1) = \frac{d}{dz}\left[\frac{1}{1 - \frac{1}{2}z}\right]\Bigg|_{z=0} = \frac{1}{2}$$

$$x(-2) = \text{Res}\left[\frac{1}{z^3(1 - \frac{1}{2}z)} \quad \text{at } z = 0\right]$$

$$= \frac{1}{2}\frac{d^2}{dz^2}\left[\frac{1}{1 - \frac{1}{2}z}\right]\Bigg|_{z=0} = \frac{1}{4}$$

(b) For $n > 0$ $x(n) = 0$ since there are no poles inside the region of convergence. For $n \leq 0$

$$x(n) = \frac{1}{2\pi j} \oint_{c'} \frac{1}{(1 - \frac{1}{2}p^{-1})}\, p^{-n-1}\, dp$$

where $|p| > \frac{1}{2}$ on c'. Thus for $n \leq 0$

$$x(n) = (\tfrac{1}{2})^{-n}$$

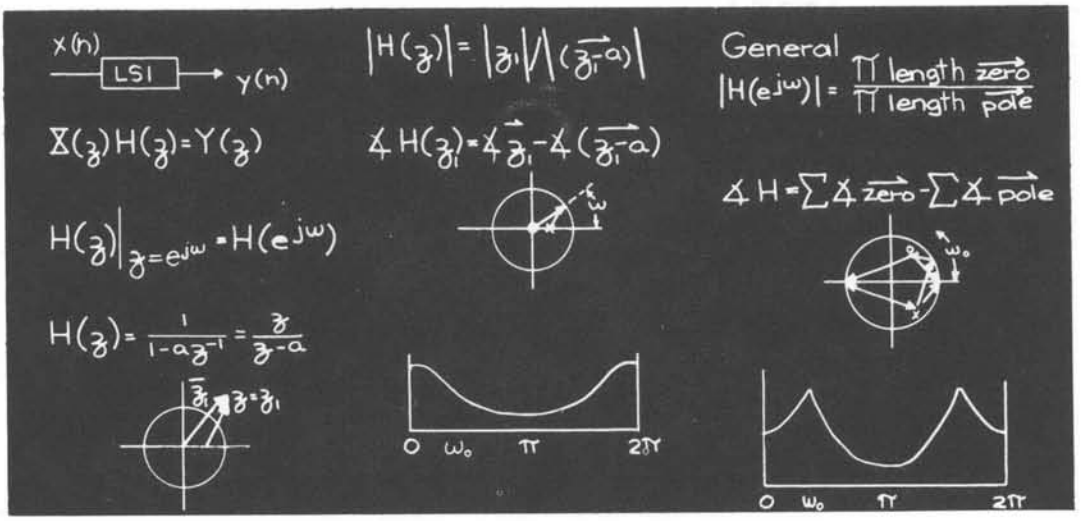
Note in particular that

$x(0) = 1$, $x(-1) = \frac{1}{2}$ and $x(-2) = \frac{1}{4}$ as was obtained in (a).

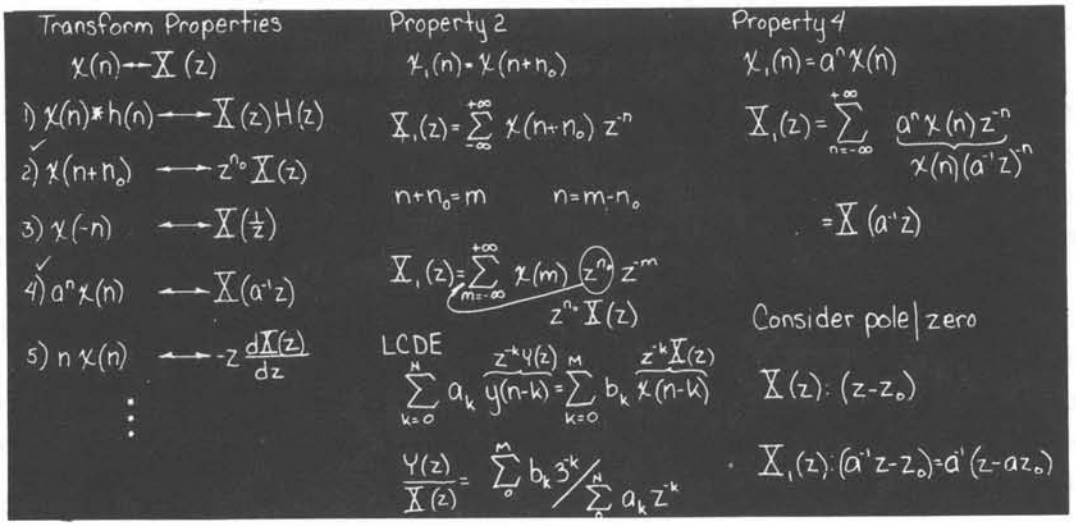
Z-TRANSFORM PROPERTIES

1. Lecture 7 - 56 minutes

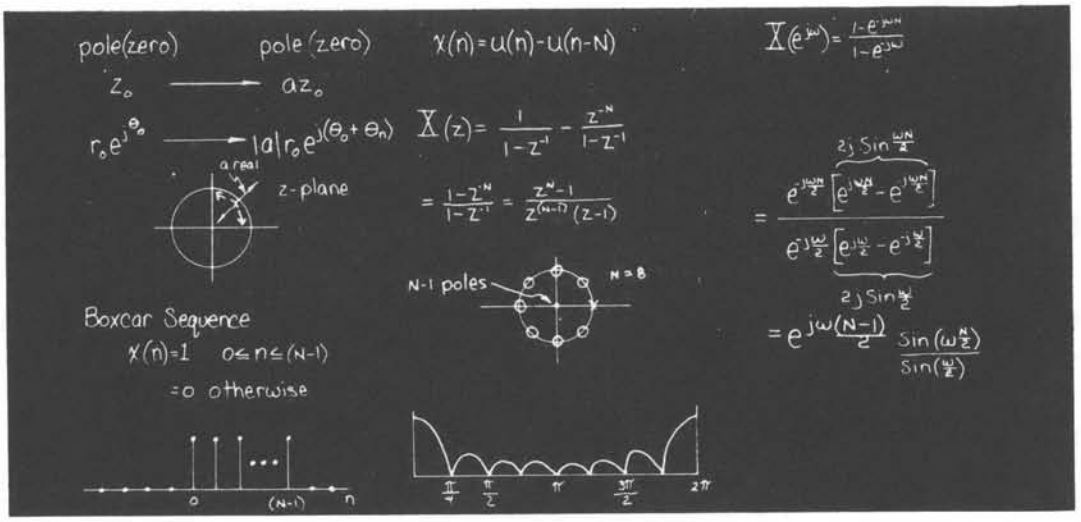
a.

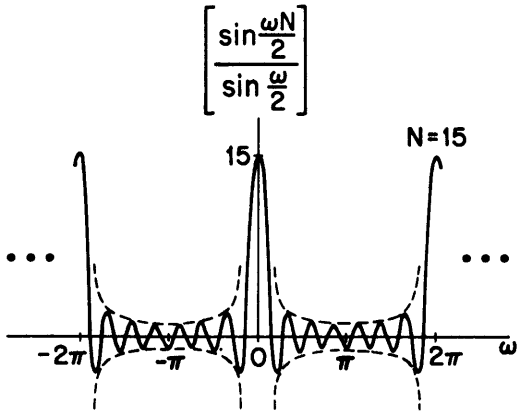


b.


Transform Properties
x(n) ↔ X(z)
1) x(n)*h(n) ↔ X(z)H(z)
2) x(n+n0) ↔ z^n0 X(z)
3) x(-n) ↔ X(1/z)
4) a^n x(n) ↔ X(a^-1 z)
5) n x(n) ↔ -z dX(z)/dz
Property 2
x1(n) = x(n+n0)
X1(z) = ∑ x(n+n0) z^-n
n+n0 = m n = m-n0
X1(z) = z^n0 X(z)
LCDE
∑ ak y(n-k) = ∑ bk x(n-k)
Y(z)/X(z) = ∑ bk z^-k / ∑ ak z^-k
Property 4
x1(n) = a^n x(n)
X1(z) = ∑ a^n x(n) z^-n / x(n)(a^-1 z)^-n
= X(a^-1 z)
Consider pole|zero
X(z): (z-z0)
X1(z): (a^-1 z - z0) = a^-1 (z - a z0)

c.





Fourier Transform of a rectangular sequence.

d.

2. Comments

In this lecture two primary ideas are discussed. The first is the determination of frequency response geometrically in the z-plane. This is particularly useful for identifying approximately the effect of pole and zero locations on the frequency response of a system. The second topic discussed in this lecture is that of properties of the z-transform. As with the Fourier transform, properties of the z-transform are useful for evaluating the z-transform and inverse z-transform as well as for developing insight into the relationship between sequences and their z-transforms. The more important properties are summarized in section 4.4 and table 4.2 (page 180) of the text. As stressed during the lecture the style in which many of these properties are proven is similar and it is important to develop a facility with this style rather than memorizing the various properties.

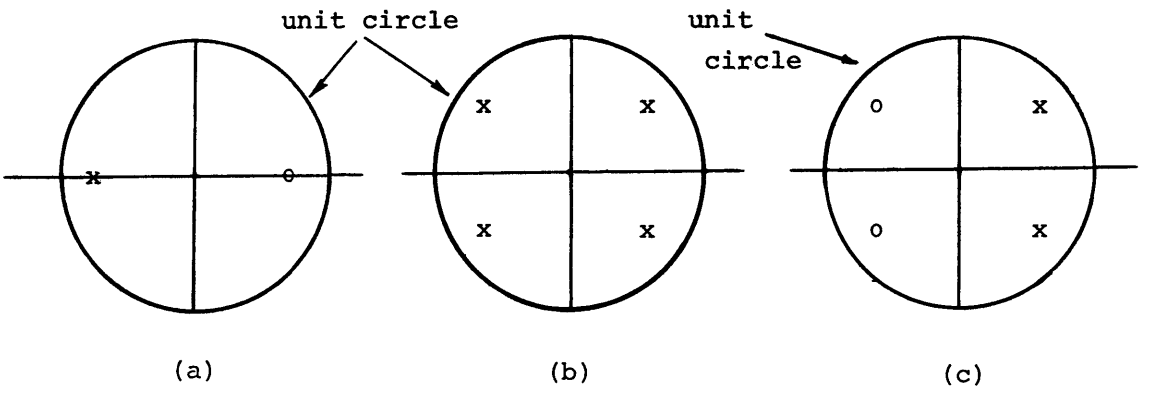
3. Reading

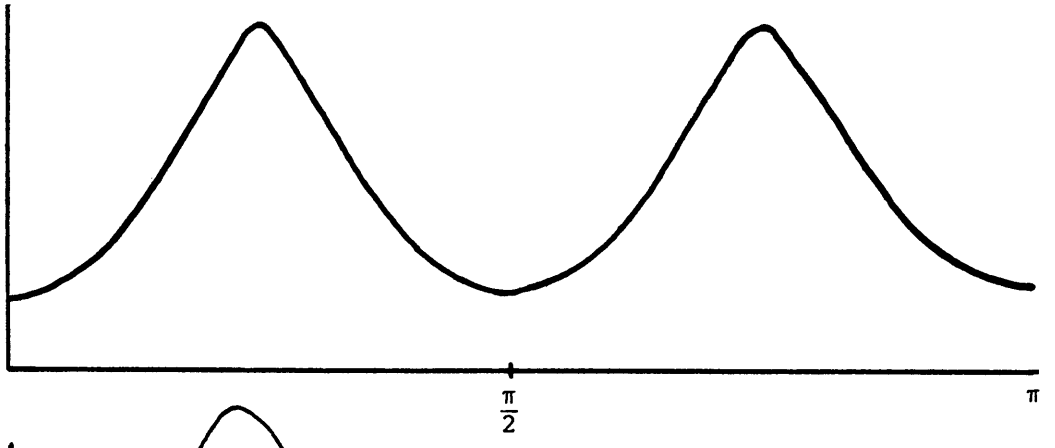
Text: Sections 4.4 (page 172), 5.2 (page 206) and 5.3.

4. Problems

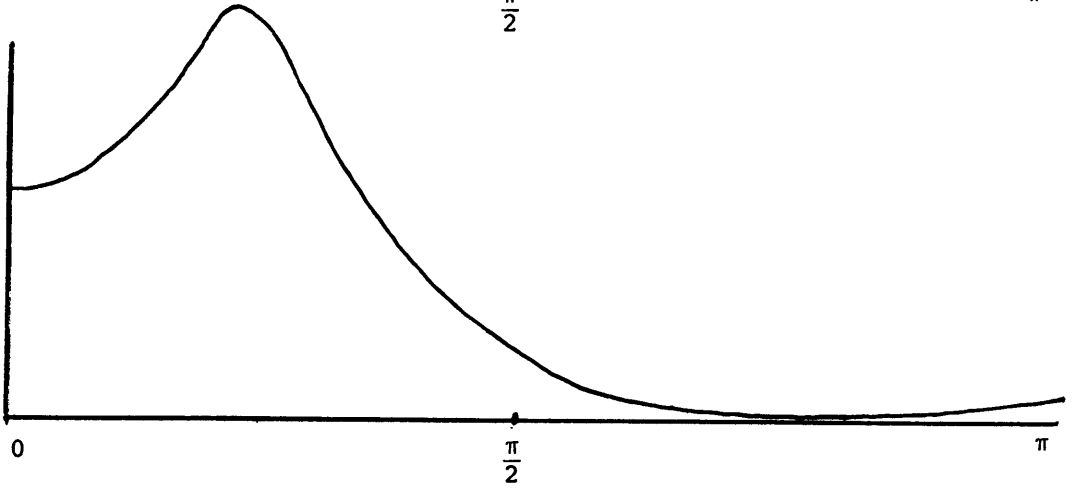
Problem 7.1

In Figure P7.1-1 are shown three pole-zero patterns and three possible frequency response magnitude characteristics. By considering the behavior of the pole and zero vectors in the z-plane determine which frequency response characteristic could correspond to each of the pole-zero patterns.

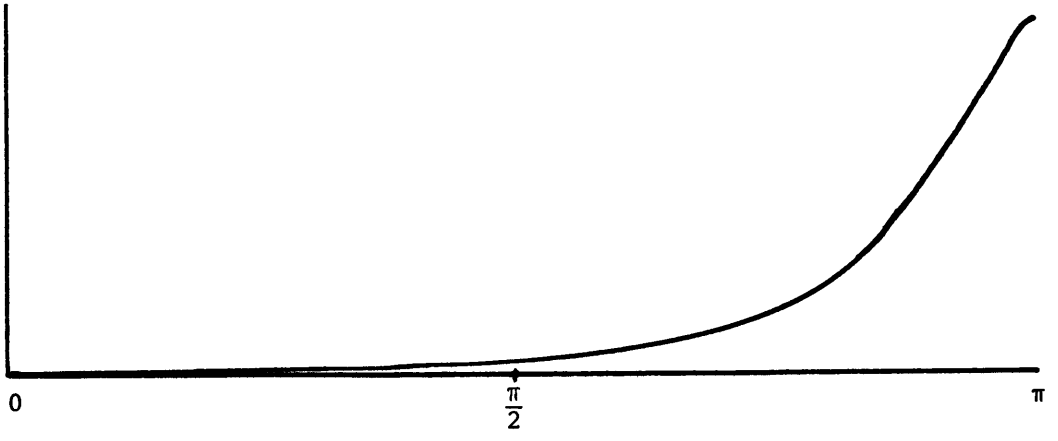




(i)



(ii)



(iii)

Figure P7.1-1

Problem 7.2

Consider a causal linear shift-invariant system with system function

$$H(z) = \frac{1 - a^{-1} z^{-1}}{1 - a z^{-1}}$$

where "a" is real

(a) If $0 < a < 1$, plot the pole-zero diagram and shade the region of convergence.

(b) Show graphically in the z-plane that this system is an allpass system, i.e., that the magnitude of the frequency response is a constant.

Problem 7.3

With $X(z)$ denoting the z-transform of $x(n)$ show that:

(i) $X(1/z)$ is the z-transform of $x(-n)$

(ii) $-z \frac{dX(z)}{dz}$ is the z-transform of $nx(n)$

Problem 7.4

Consider a linear discrete-time shift-invariant system with input $x(n)$ and output $y(n)$ for which

$$y(n - 1) - \frac{10}{3} y(n) + y(n + 1) = x(n)$$

The system is stable. Determine the unit-sample response.

Problem 7.5

$$X(z) = \log_e (1 - az^{-1})$$

where $|a| < 1$ and the region of convergence is $|z| > |a|$.

(a) Determine the inverse z-transform by using the differentiation property.

(b) Determine the inverse z-transform by using the power series method. Compare your result with that obtained in (a).

Problem 7.6*

Suppose that we have a sequence $x(n)$ from which we construct a new sequence $x_1(n)$ defined as

$$x_1(n) = \begin{cases} x(\frac{n}{M}) & n = 0, \pm M, \pm 2M, \ldots \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine $X_1(z)$ in terms of $X(z)$

(b) If $X(e^{j\omega})$ is as sketched below, sketch $X_1(e^{j\omega})$ for M = 2.

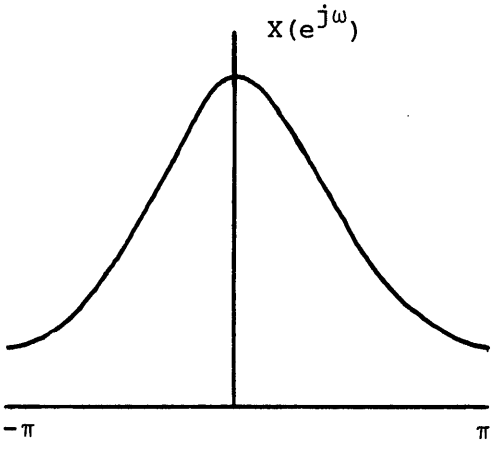


Figure P7.6-1

Z-TRANSFORM PROPERTIES

Solution 7.1

(a) - (iii)
(b) - (i)
(c) - (ii)

Solution 7.2

(a)

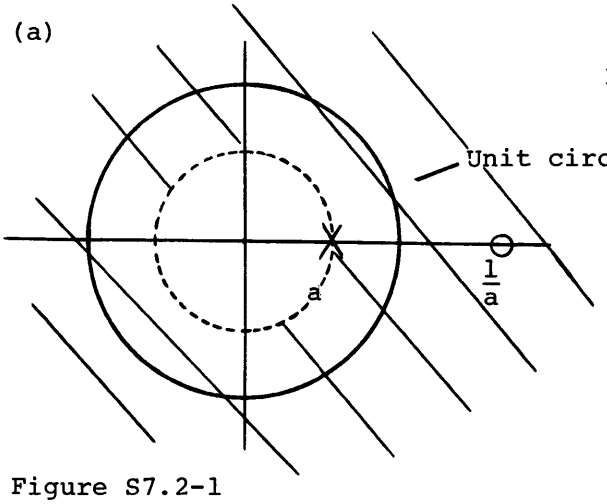


Figure S7.2-1

(b)

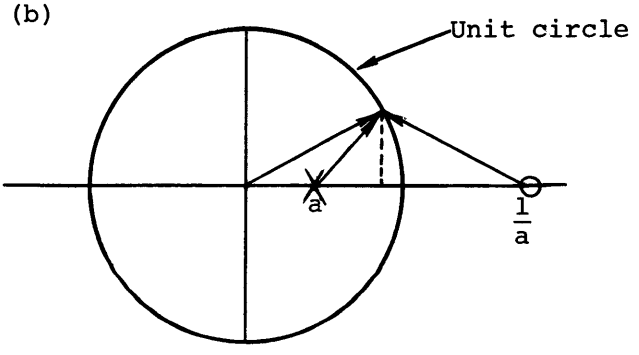


Figure S7.2-2

Shown above is the appropriate vector diagram, which we expand out below:

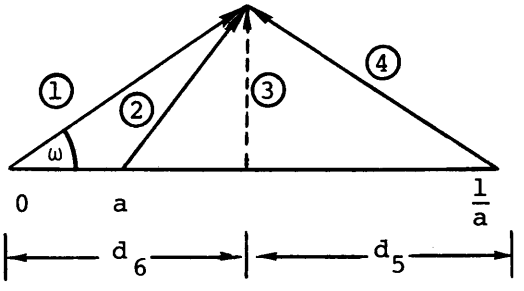


Figure S7.2-3

Let d_1, d_2, d_3 and d_4 denote the lengths of vectors ①, ②, ③ and ④ respectively. We wish to determine $\frac{d_4}{d_2}$. We know that vector ① is at an angle of ω to the horizontal axis and that $d_1 = 1$. Therefore:

$$d_6 = \cos \omega$$

$$d_3 = \sin \omega$$

$$d_5 = \frac{1}{a} - d_6 = \frac{1}{a} - \cos \omega$$

then

$$d_2^2 = (d_6 - a)^2 + d_3^2 = 1 + a^2 - 2a \cos \omega$$

Also,

$$d_4^2 = d_3^2 + d_5^2 = 1 + (\frac{1}{a})^2 - \frac{2}{a} \cos \omega = (\frac{1}{a})^2 \left[1 + a^2 - 2a \cos \omega\right]$$

Thus

$$\frac{d_4}{d_2} = \frac{1}{a}$$

Solution 7.3

(i)

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n}$$

$$X(\frac{1}{z}) = \sum_{n=-\infty}^{+\infty} x(n) z^{n}$$

With the substitution of variables $m = -n$ in the above summation,

$$X(\frac{1}{z}) = \sum_{m=-\infty}^{+\infty} x(-m) z^{-m}$$

which we recognize as the z-transform of x(-n).

(ii)

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n}$$

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{+\infty} -nx(n) z^{-n-1}$$

$$-z \frac{dX(z)}{dz} = \sum_{n=-\infty}^{+\infty} nx(n) z^{-n}$$

We recognize the right-hand side as the z-transform of n x(n).

Solution 7.4

Letting X(z) and Y(z) denote the z-transforms of x(n) and y(n), and using the properties of z-transforms, the z-transform of the difference equation results in

$$z^{-1} Y(z) - \frac{10}{3} Y(z) + z\, Y(z) = X(z)$$

thus the system function H(z) is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{z^{-1} - \frac{10}{3} + z} = \frac{z^{-1}}{z^{-2} - \frac{10}{3}z^{-1} + 1}$$

$$= \frac{z^{-1}}{(1 - 3z^{-1})(1 - \frac{1}{3}z^{-1})}$$

To determine the unit-sample response we can obtain the inverse transform of H(z) using any of the methods that we have discussed. For example, using contour integration,

$$x(n) = \frac{1}{2\pi j} \oint_C \frac{z^{-1}}{(1 - 3z^{-1})(1 - \frac{1}{3}z^{-1})} z^{n-1}\, dz$$

Since the system is stable, the region of convergence includes the unit circle. Thus for $n \geq 0$

$$x(n) = \text{Res}\left[\frac{z^n}{(z - 3)(z - \frac{1}{3})} \text{ at } z = \frac{1}{3}\right] = -\frac{3}{8}\left(\frac{1}{3}\right)^n$$

For $n < 0$

$$x(n) = \frac{1}{2\pi j} \oint_C \frac{p}{(1 - 3p)(1 - \frac{1}{3}p)} p^{-n-1}\, dp$$

$$= \text{Res}\left[\frac{p^{-n}}{(1 - \frac{9}{3}p)(1 - \frac{1}{3}p)} \text{ at } p = \frac{1}{3}\right] = -\frac{3}{8}\left(\frac{1}{3}\right)^{-n}$$

therefore,

$$x(n) = -\frac{3}{8}\left[\left(\frac{1}{3}\right)^n u(n) + (3)^n u(-n-1)\right]$$

Solution 7.5

(a) According to the differentiation property, the z-transform of nx(n) is $-z\frac{dX(z)}{dz}$. For this problem,

$$-z\frac{dX(z)}{dz} = -\frac{az^{-1}}{(1-az^{-1})}$$

The inverse z-transform of $-\frac{a}{(1-az^{-1})}$ is $-a(a)^n u(n)$. From the shifting property, then, the inverse z-transform of $\frac{-az^{-1}}{(1-az^{-1})}$ is $-a(a)^{n-1}u(n-1)$.

Therefore

$$n\,x(n) = -a^n u(n-1)$$

or

$$x(n) = -\frac{a^n}{n}u(n-1) \qquad n \neq 0$$

Note that since we obtain $nx(n)$ from the differentiation property, this does not allow us to obtain $x(0)$. For this problem, however, we could obtain $x(0)$ from problem 5.7. Specifically, since $x(n)$ is causal,

$$x(0) = \lim_{z\to\infty} \log(1-az^{-1}) = 0$$

(b) For $|\rho| < 1$, the power series expansion for $\log(1-\rho)$ is

$$\log(1-\rho) = -\sum_{n=1}^{\infty}\frac{\rho^n}{n}$$

thus,

$$\log(1-az^{-1}) = -\sum_{n=1}^{\infty}\frac{a^n}{n}z^{-n}$$

thus we identify $x(n)$ as

$$x(n) = -\frac{a^n}{n}u(n-1)$$

Note this problem is strongly related to the discussion in chapter 12 of the text. You may wish to look through some of the discussion in that chapter.

Solution 7.6

$$X_1(z) = \sum_{n=-\infty}^{+\infty} x_1(n)z^{-n} = \sum_{n=-\infty}^{+\infty} x(n)z^{-nM}$$

Therefore $X_1(z) = X(z^M)$

For $M = 2$

$$X_1(e^{j\omega}) = X(e^{j2\omega})$$

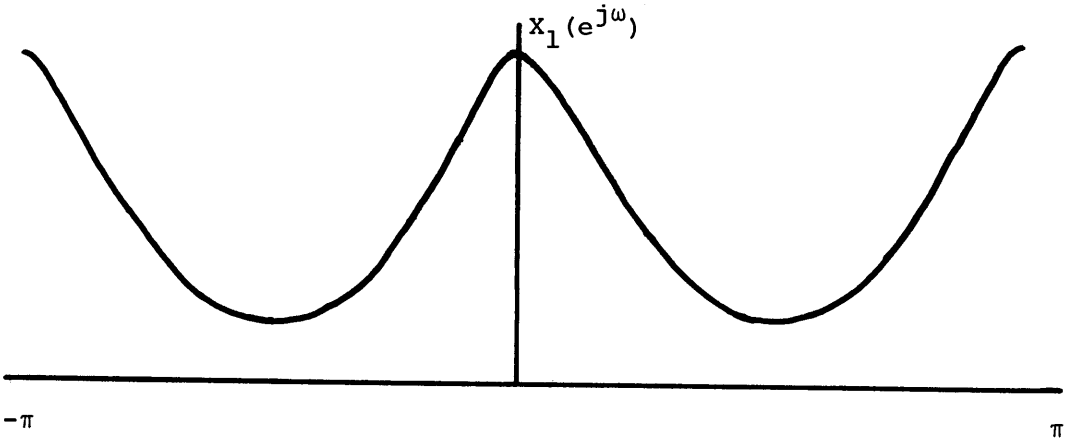


Figure S7.6-1

THE DISCRETE FOURIER SERIES

1. Lecture 8 - 43 minutes

a.

$x(n)$ Finite length

$x(n) = \tilde{x}(n) \quad 0 \le n \le (N-1)$

$= 0$ otherwise

$\tilde{x}(n)$ has a Fourier Series Representation

$\tilde{x}(n)$ Periodic

$\tilde{x}(n) = x(n) + x(n+N) + \ldots$

$x(n) = \tilde{x}(n)\, R_N(n)$

DFS of $\tilde{x}(n)$ $\triangleq$ DFT of $x(n)$

$R_N(n) = 1 \quad 0 \le n \le (N-1)$

$= 0$ otherwise

Discrete Fourier Series

$\tilde{x}(n)$: periodic, period N

$\tilde{x}(n) = \sum \tilde{X}(k) e^{j\frac{2\pi}{N}nk}$

b.

$e^{j\frac{2\pi}{N}nk} = \underbrace{e^{j\frac{2\pi}{N}n(k+N)}}_{e^{j\frac{2\pi}{N}nk}\, e^{j\frac{2\pi}{N}n\cdot N}\,(=1)}$

$W_N \triangleq e^{-j\frac{2\pi}{N}}$

Symmetry: $\tilde{x}(n)$ real

$\tilde{x}(n) = \frac{1}{N}\sum_{K=0}^{N-1} \tilde{X}(k) e^{j\frac{2\pi}{N}nk}$

$\tilde{x}(n) = \frac{1}{N}\sum_{K=0}^{N-1} \tilde{X}(k) W_N^{-nk}$

$\tilde{X}(k) = \tilde{X}_R(k) + j\tilde{X}_I(k)$

$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n) e^{-j\frac{2\pi}{N}nk}$

$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n) W_N^{nk}$

$\tilde{X}_R(k) = \tilde{X}_R(-k)$ even

$= \tilde{X}_R(N-k)$

DFS Properties

$\tilde{X}_I(k) = -\tilde{X}_I(-k)$ odd

$= -\tilde{X}_I(N-k)$

$e^{-j\frac{2\pi}{N}n(K+N)} = e^{-j\frac{2\pi}{N}nk}$

Shifting

$\tilde{x}(n+m) \quad W_N^{-km}\tilde{X}(k)$

$|\tilde{X}(k)|$ even

$\tilde{X}(k)$ periodic in k

period N

$W_N^{\ell n}\tilde{x}(n) \quad \tilde{X}(k+\ell)$

$\measuredangle \tilde{X}(k)$ odd

c.

Convolution Property

$\tilde{x}_1(n) \leftrightarrow \tilde{X}_1(k)$

$\tilde{x}_2(n) \leftrightarrow \tilde{X}_2(k)$

$\tilde{x}_3(n) \leftrightarrow \tilde{X}_1(k)\tilde{X}_2(k)$

$\tilde{x}_3(n) = \sum_{m=0}^{N-1} \tilde{x}_1(m)\tilde{x}_2(n-m)$

Dual Property

$\tilde{x}_4(n) = \tilde{x}_1(n)\tilde{x}_2(n)$

$\tilde{X}_4(k) = \frac{1}{N}\sum_{\ell=0}^{N-1} \tilde{X}_1(\ell)\tilde{X}_2(k-\ell)$

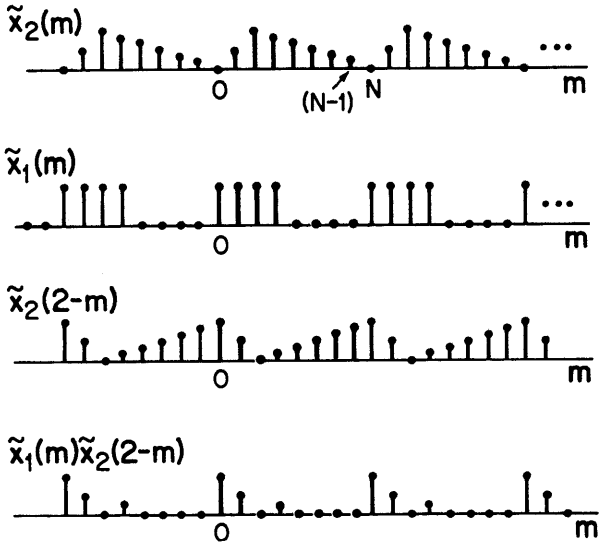


Illustration of the sequences involved in forming a periodic convolution.

d.

Correction: Note that the view-graph shown in the lecture illustrating periodic convolution does not correctly illustrate the sequence $\tilde{x}_1(m)\ \tilde{x}_2(2 - m)$. The copy above of this figure includes the correction

2. Comments

In the past several lectures we considered the Fourier transform and z-transform as tools for the representation and analysis of discrete-time signals and systems. In this lecture we introduce the Fourier series representation for periodic sequences. In the next lecture the Fourier series will be applied to developing a Fourier representation of finite length sequences, referred to as the Discrete Fourier Transform by utilizing the fact that finite length and periodic sequences are closely related. Because of the importance of this relationship in eventually interpreting properties of the Discrete Fourier Transform, this lecture begins with a discussion of the relationship betweeen finite length and periodic sequences. The remainder of the lecture concentrates on the Fourier series representation of periodic sequences.

In representing continuous-time signals through the Fourier series, the representation in general requires an infinite number of harmonically related complex exponentials (or sines and cosines). In contrast, in the discrete-time case there are only a finite number (N) of distinguishable harmonically related complex exponentials with fundamental period N. Thus there are only a finite number (N) of distinguishable Discrete Fourier Series coefficients. In developing the Fourier series representation it is convenient to interpret the Fourier series coefficients as a periodic sequence with period N. This interpretation

is consistent with the property that there are only N distinguishable coefficients since the distinguishable values in a periodic sequence are represented by a single period.

The lecture concludes with a discussion of some of the properties of Fourier series coefficients. A property which will play an important role in lecture 10 is the convolution property, and in particular the definition of periodic convolution.

3. Reading

Text: Sections 8.0 (page 514) through 8.3.

4. Problems

Problem 8.1

(a) The sequence $\tilde{x}_1(n)$ of Figure P8.1-1 is periodic with period 4. Determine the Fourier series coefficients $\tilde{X}_1(k)$

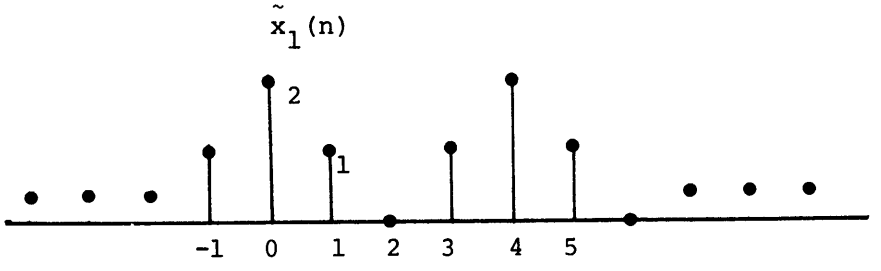


Figure P8.1-1

Problem 8.2

In the lecture it was stated that for a real periodic sequence $\tilde{x}(n)$ the Fourier series coefficients $\tilde{X}(k)$ are conjugate symmetric, i.e. $\tilde{X}(k) = \tilde{X}^*(-k)$. By using the Discrete Fourier Series analysis and synthesis pair, (eqs 8.11 and 8.12 in text) show that this symmetry property is true.

Problem 8.3

Another important symmetry property of the Fourier series coefficients states that if $\tilde{x}(n)$ is real and even, then $\tilde{X}(k)$ is real and even. By using the Discrete Fourier Series analysis and synthesis pair show that the symmetry property is true.

Problem 8.4

In figure P8.4-1 is shown a real periodic signal $\tilde{x}(n)$. Utilizing the properties discussed in section 8.2 and without explicitly evaluating the Fourier series coefficients, determine whether or not the following are true or not for the Fourier series coefficients $\tilde{X}(k)$.

(i) $\tilde{X}(k) = \tilde{X}(k + 10)$ for all k

(ii) $\tilde{X}(k) = \tilde{X}(-k)$ for all k

(iii) $\tilde{X}(0) = 0$

(iv) $\tilde{X}(k)\, e^{jk\frac{2\pi}{5}}$ is real for all k

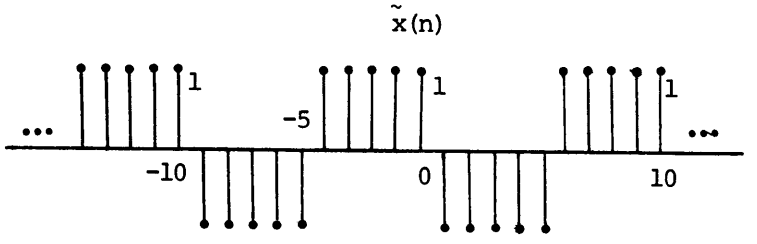


Figure P8.4-1

Problem 8.5

In figure P8.5-1 are shown two periodic sequences both with period 6. Determine and sketch $\tilde{x}_3(n)$ the result of a periodic convolution of these two sequences.

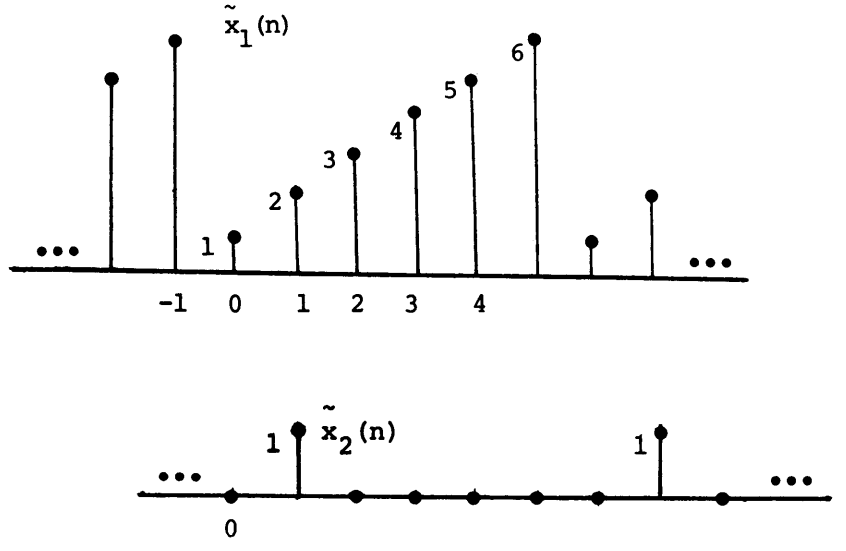


Figure P8.5-1

Problem 8.6 *

$\tilde{x}(n)$ denotes a periodic sequence with period N and $\tilde{X}(k)$ denotes its discrete Fourier series coefficients. The sequence $\tilde{X}(k)$ is also a periodic sequence with period N. Determine, in terms of $\tilde{x}(n)$, the discrete Fourier series coefficients of $\tilde{X}(k)$.

Problem 8.7 *

In Figure P8.7-1 are shown several periodic sequences x(n). These sequences can be expressed in a Fourier Series as

$$\tilde{x}(n) = \sum_{k=0}^{N-1} X(k)\ e^{j(2\pi/N)\ kn}$$

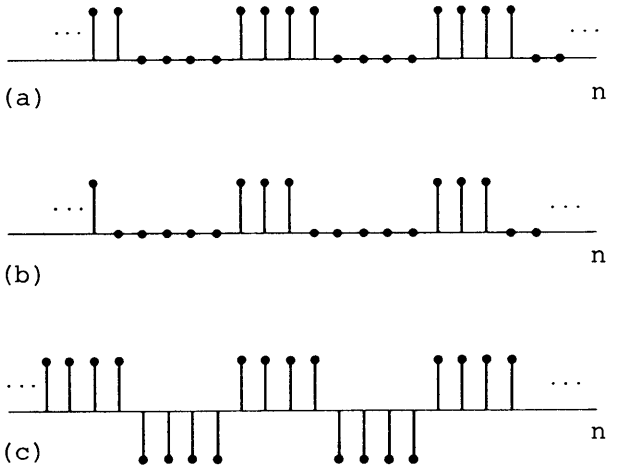


Figure P8.7-1

(a) For which sequences can the time origin be chosen such that all the X(k) are real?

(b) For which sequences can the time origin be chosen such that all the X(k) (except X(0)) are imaginary?

Problem 8.8 *

If $\tilde{x}(n)$ is a periodic sequence with period N, it is also periodic with period 2N. Let $\tilde{X}_1(k)$ denote the DFS coefficients of $\tilde{x}(n)$ considered as a periodic sequence with period N and $\tilde{X}_2(k)$ denote the DFS coefficients of $\tilde{x}(n)$ considered as a periodic sequence with period 2N. $\tilde{X}_1(k)$ is of course, periodic with period N and $\tilde{X}_2(k)$ is periodic with period 2N. Determine $\tilde{X}_2(k)$ in terms of $\tilde{X}_1(k)$.

THE DISCRETE FOURIER SERIES

Solution 8.1

$$\tilde{X}_1(k) = \sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j\frac{2\pi}{N}nk}$$

$$= 2 + e^{-j\frac{\pi}{2}k} + e^{-j\frac{\pi}{2}3k}$$

$$e^{-j\frac{3\pi}{2}k} = e^{-jk(-\frac{3\pi}{2}+2\pi)} = e^{j\frac{\pi}{2}k}$$

Therefore,

$$\tilde{X}_1(k) = 2 + e^{-j\frac{\pi}{2}k} + e^{j\frac{\pi}{2}k} = 2\left[1 + \cos\frac{\pi k}{2}\right].$$

Solution 8.2

$$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n)\, W_N^{kn}$$

$$\tilde{X}^*(k) = \sum_{n=0}^{N-1} \tilde{x}^*(n)\, W_N^{-kn}$$

or, since $\tilde{x}(n)$ is real,

$$\tilde{X}^*(k) = \sum_{n=0}^{N-1} \tilde{x}(n)\, W_N^{-kn}$$

Finally, substituting -k for k

$$\tilde{X}^*(-k) = \sum_{n=0}^{N-1} \tilde{x}(n)\, W_N^{kn} = \tilde{X}(k)$$

Note, incidentally, that this is indeed satisfied for problem 8.1.

Solution 8.3

If we show that $\tilde{X}(k)$ is real, then from problem 8.2 it follows that $\tilde{X}(k)$ is also even. Thus

$$\tilde{X}^*(k) = \sum_{n=0}^{N-1} \tilde{x}(n)\, W_N^{-kn}$$

Replacing n by -n in the summation on the right-hand side

$$\tilde{X}^*(k) = \sum_{n=0}^{-N+1} \tilde{x}(-n)\, W_N^{kn}$$

or since $\tilde{x}(n)$ is even

$$\tilde{X}^*(k) = \sum_{n=0}^{-N+1} \tilde{x}(n)\, W_N^{kn}$$

Finally, since $\tilde{x}(n)$ is periodic the limits on the summation can be replaced by the interval 0 to N-1. Thus $\tilde{X}^*(k) = \tilde{X}(k)$, i.e. $\tilde{X}(k)$ is real.

Solution 8.4

(i) Since $\tilde{x}(n)$ is periodic with period 10, $\tilde{X}(k)$ is also periodic with period 10. Thus (i) is true.

(ii) Since $\tilde{x}(n)$ is real, $\tilde{X}^*(k) = \tilde{X}(-k)$. In order for the stated property to also be true, $\tilde{X}(k)$ must be real, which requires that $\tilde{x}(n)$ be even, which is not the case. Thus (ii) is not true.

(iii) $\tilde{X}(0) = \sum_{n=0}^{N-1} \tilde{x}(n) = 0$. Thus (iii) is true.

(iv) $\tilde{X}(k)\, e^{jk\frac{2\pi}{5}}$ is the Fourier series for $\tilde{x}(n + 2)$. From the figure we note that $\tilde{x}(n + 2)$ is not an even function. Thus $\tilde{X}(k)\, e^{jk\frac{2\pi}{5}}$ is not real. However, $\tilde{x}(n - 2)$ is an even sequence and thus $\tilde{X}(k)\, e^{-jk\frac{2\pi}{5}}$ is real

Solution 8.5

See Figure S8.5-1 on next page.

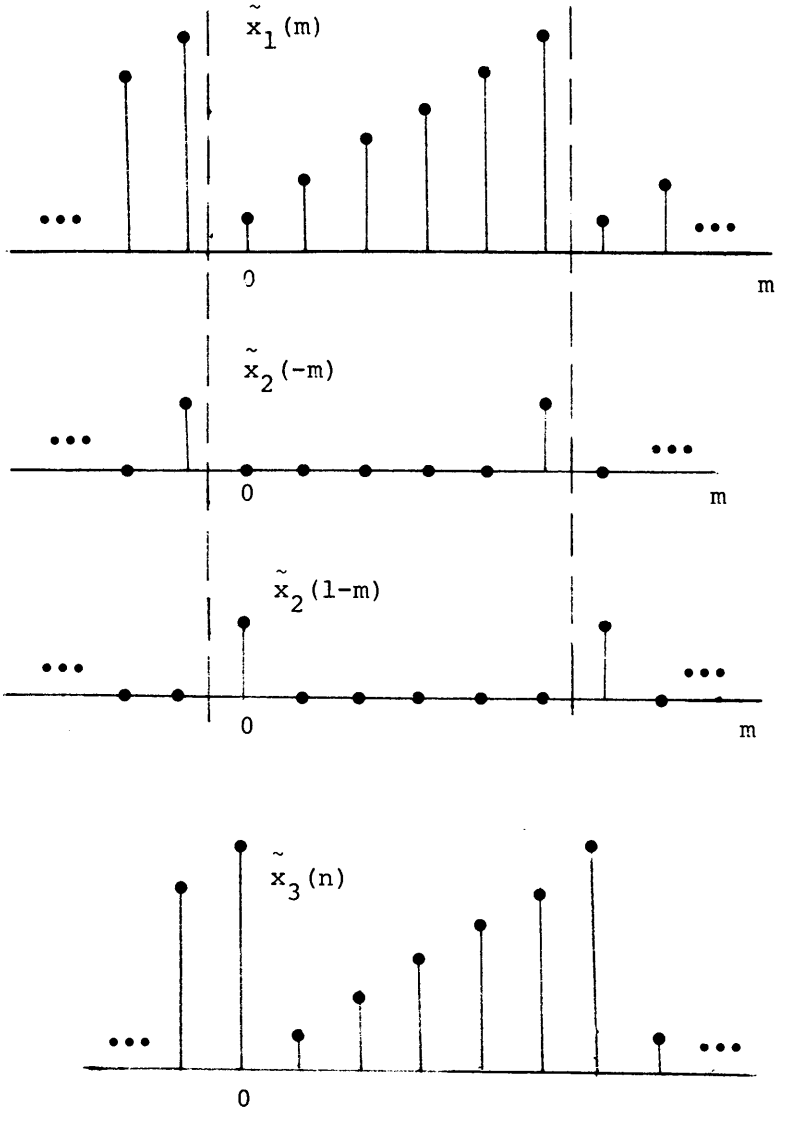


Figure S8.5-1

Solution 8.6

The Discrete Fourier series coefficients of $\tilde{X}(k)$ would be defined as

$$\tilde{y}(n) = \sum_{k=0}^{N-1} \tilde{X}(k) W_N^{kn}$$

$\tilde{x}(n)$ is given by

$$\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) \; W_N^{-kn}$$

thus $\tilde{y}(n) = N \; \tilde{x}(-n)$.

Solution 8.7

(a) The time origin can be chosen such that all the $\tilde{X}(k)$ are real if $x(n)$ can be shifted to be an even function. It can for sequence (b) but not for the others.

(b) This requires that the time origin be chosen so that $\tilde{x}(n)$ is odd. This cannot be done for any of the sequences.

Solution 8.8

$$\tilde{X}_1(k) = \sum_{n=0}^{N-1} \tilde{x}(n)\, W_N^{k}$$

$$\tilde{X}_2(k) = \sum_{n=0}^{2N-1} \tilde{x}(n)\, W_{2N}^{kn}$$

$$= \sum_{n=0}^{N-1} \tilde{x}(n)\, W_{2N}^{kn} + \sum_{n=0}^{N-1} \tilde{x}(n+N)\, W_{2N}^{k(n+N)}$$

or, since $\tilde{x}(n)$ is periodic with period N and $W_{2N}^{N} = -1$

$$\tilde{X}_2(k) = \sum_{n=0}^{N-1} \tilde{x}(n)\, W_{2N}^{kn}\, [1 + (-1)^k]$$

$$= [1+(-1)^k] \sum_{n=0}^{N-1} \tilde{x}(n)\, W_{2N}^{kn}$$

Thus, for k odd, $\tilde{X}_2(k) = 0$. For k even, $W_{2N}^{kn} = W_N^{n(k/2)}$

and

$$\tilde{X}_2(k) = 2 \sum_{n=0}^{N-1} \tilde{x}(n)\, W_N^{n(k/2)}$$

$$= 2\, \tilde{X}_1(k/2) \qquad k \text{ even.}$$

THE DISCRETE FOURIER TRANSFORM

1. Lecture 9 - 42 minutes

a.

$x(n)=0 \quad n<0,\ n>(N-1)$

finite length N (or less)

$\tilde{x}(n)=\sum_{r=-\infty}^{+\infty} x(n+rN)$

$= x(n \text{ modulo } N) \triangleq x((n))_N$

$x(n)=\tilde{x}(n)\,R_N(n)$

$\tilde{X}(k)=$ DFS of $\tilde{x}(n)$

Discrete Fourier Series

$\tilde{X}(k)=\sum_{n=0}^{N-1}\tilde{x}(n)\,W_N^{nk}$

$\tilde{x}(n)=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}(k)\,W_N^{-nk}$

Discrete Fourier Transform

$X(k)=\sum_{n=0}^{N-1}x(n)\,W_N^{nk} \quad k=0,1,\ldots,N-1$

$=0$ otherwise

$X(k)=\tilde{X}(k)\,R_N(k)$

$\tilde{X}(k)=X((k))_N$

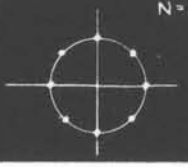
$X(k)=\left[\sum_{n=0}^{N-1}x(n)W_N^{nk}\right]R_N(k)$

$x(n)=\left[\frac{1}{N}\sum_{k=0}^{N-1}X(k)W_N^{-nk}\right]R_N(n)$

$X(z)=\sum_{n=0}^{N-1}x(n)\,z^{-n}$

$X(k)=X(z)\big|_{z=W_N^{-k}} \quad k=0,1,\ldots,N-1$

N=8



b.

Properties of the DFT

Shifting Property

$x(n) \longleftrightarrow X(k)$

$\tilde{x}(n) \longleftrightarrow \tilde{X}(k)$

$\tilde{x}_1(n)=\tilde{x}(n+m) \longleftrightarrow \tilde{X}(k)W_N^{-km}$

$x_1(n) \longleftrightarrow X(k)W_N^{-km}$

$x((n+m))_N R_N(n) \longleftrightarrow W_N^{-km}X(k)$

$W_N^{\ell n}x(n) \longleftrightarrow X((k+\ell))_N R_N(k)$

Symmetry Properties

DFS $\tilde{x}(n)$ real

$\tilde{X}_R(k)=\tilde{X}_R(N-k)$

$\tilde{X}_I(k)=-\tilde{X}_I(N-k)$

DFT $x(n)$ real

$X_R(k)=X_R((N-k))_N R_N(k)$ (even N)

$X_I(k)=-X_I((N-k))_N R_N(k)$ (odd)

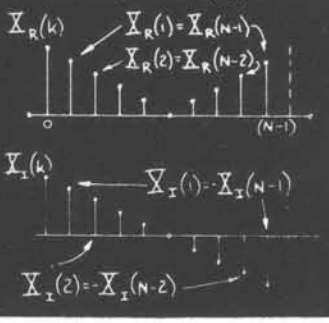
for example

$X_R(0)$

$X_R(0)=X_R((N-0))_N R_N(0)$ (1)

$X_R(N-1)$

$X_R(1)=X_R((N-1))_N R_N(1)$ (1)



c.

Convolution Property

$x_3(n) \longleftrightarrow X_1(k)\,X_2(k)$

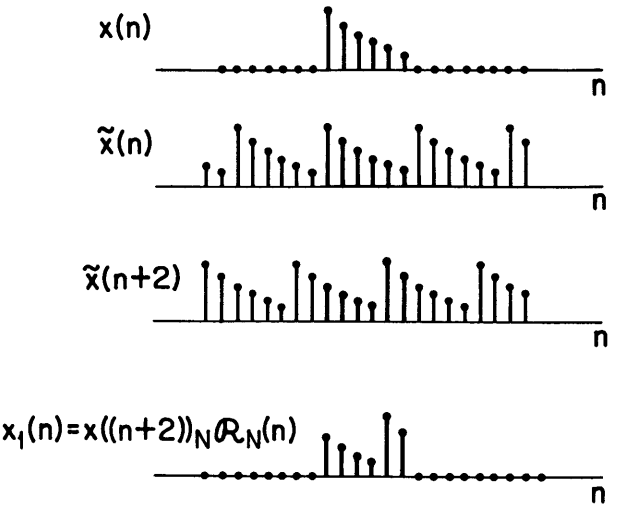
$\tilde{x}_3(n) \longleftrightarrow \tilde{X}_1(k)\,\tilde{X}_2(k)$

$x_3(n)=\tilde{x}_3(n)\,R_N(n)$

$x_3(n)=\left[\sum_{m=0}^{N-1}\tilde{x}_1(m)\,\tilde{x}_2(n-m)\right]R_N(n)$

$=\left[\sum_{m=0}^{N-1}x_1((m))_N\,x_2((n-m))_N\right]R_N(n)$

$x_3(n)=x_1(n)\ \textcircled{N}\ x_2(n)$



Circular shifting of a finite length sequence.

d.

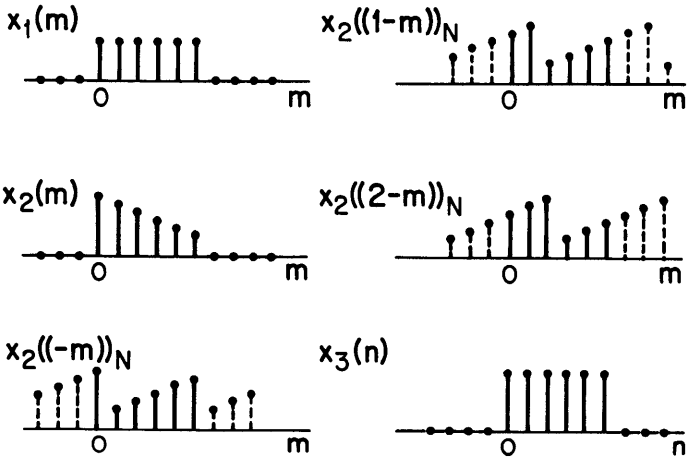


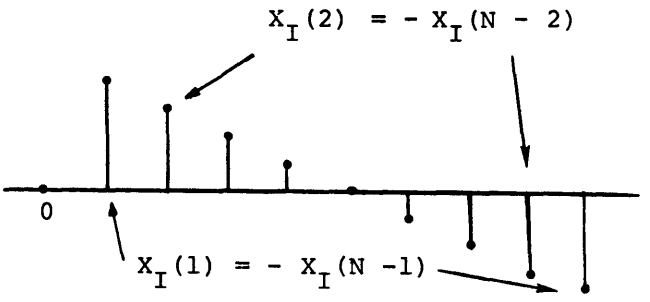
Illustration of circular convolution. (Note that $x_2((-m))_N$ is incorrectly drawn. In Problem 9.4 you are asked to correct this.)

e.

2. Corrections

Two of the figures used in this lecture require corrections. The first is the figure on board 2 which illustrates the symmetry for the imaginary part of the DFT. The value indicated as a dashed line should have a height and polarity identical to $X_I(0)$ since it represents the periodic extension of $X_I(k)$. Furthermore, because of the symmetry imposed on $X_I(k)$ the value $X_I(0)$ is constrained to be zero (see Problem 9.3 below.) A corrected version is shown at the top of page 9.3.

The second figure requiring correction is the viewgraph used to illustrate circular convolution. On that figure $x_2((-m))_N$ is incorrectly drawn. In problem 9.4 below you are asked to correct this figure.



3. Comments

In this lecture the representation of finite length sequences by means of the Discrete Fourier Transform is introduced. This representation corresponds essentially to the Discrete Fourier Series representation of the periodic counterpart of the finite length sequence.

The relationship between the DFT and the z-transform is also stressed, deriving in particular the fact that the DFT corresponds to samples of the z-transform equally spaced in angle around the unit circle. This also implies, of course, that the Fourier series coefficients of a periodic sequence correspond to samples on the unit circle of the z-transform of one period of the periodic sequence.

The lecture concludes with a discussion of properties of the DFT. All of the properties of the DFT are direct consequences of the properties of Fourier series and are generally different than the properties of the Fourier transform as discussed in lecture 4. The shifting property, for example relates to a circular shift of a sequence rather than a linear shift, and it is a circular convolution that results in a product of DFT's.

4. Reading

Text: Sections 8.5 (page 527), 8.6 and 8.7. Section 8.7.5 on circular convolution should be read only lightly for this lesson. It will be a main focus for lesson 10.

5. Problems

Problem 9.1

Compute the DFT of each of the following finite-length sequences considered to be of length N.

(a) $x(n) = \delta(n)$.

(b) $x(n) = \delta(n - n_0)$, where $0 < n_0 < N$.

(c) $x(n) = a^n$, $0 \leq n \leq N - 1$.

Problem 9.2

In figure P9.2-1 below is shown a finite length sequence x(n). Sketch the sequences $x_1(n)$ and $x_2(n)$ specified as

$$x_1(n) = x((n - 2))_4 \; R_4(n)$$

$$x_2(n) = x((-n))_4 \; R_4(n)$$

(Note that $x_1(n)$ is x(n) circularly shifted by two points).

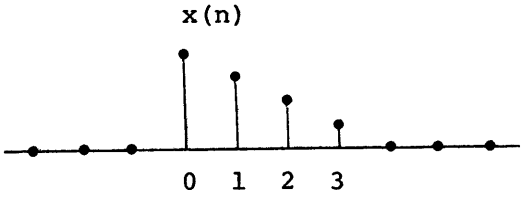


Figure P9.2-1

Problem 9.3

In the lecture we showed that X(k), the DFT of a real finite length sequence is conjugate symmetric, i.e.

$$X_R(k) = X_R((N - k))_N \; R_N(k)$$

and

$$X_I(k) = - X_I((N - k))_N \; R_N(k)$$

From this symmetry property, show that $X_I(0)$ must be zero. Also show that if N is even then $X_I(N/2)$ must be zero.

Problem 9.4

In the viewgraph used to illustrate circular convolution, the sequence $x_2((-m))_N$ is incorrectly drawn. Sketch the correct sequence $x_2((-m))_N$.

Problem 9.5

In Figure P9.5-1 below are shown two finite length sequences. Sketch their 6-point circular convolution.

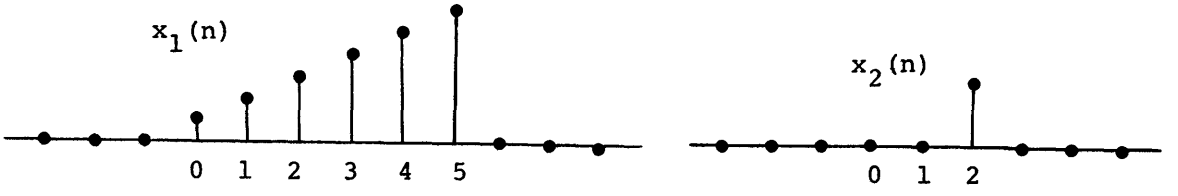


Figure P9.5-1

Problem 9.6*

The DFT of a finite-duration sequence corresponds to samples of its z-transform on the unit circle. For example, the DFT of a 10-point sequence x(n) corresponds to samples of X(z) at the 10 equally spaced points indicated in figure P9.6-1. We wish to find the equally spaced samples of X(z) on the contour shown in figure P9.6-2; i.e., $X(z)\big|_{z=0.5e^{j\left[(2\pi k/10)+(\pi/10)\right]}}$. Show how to modify x(n) to obtain a sequence $x_1(n)$ such that the DFT of $x_1(n)$ corresponds to the desired samples of X(z).

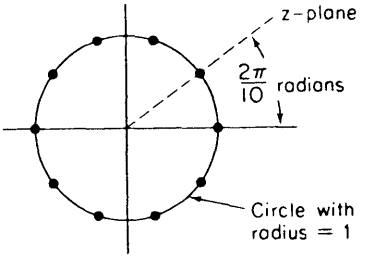


Figure P9.6-1

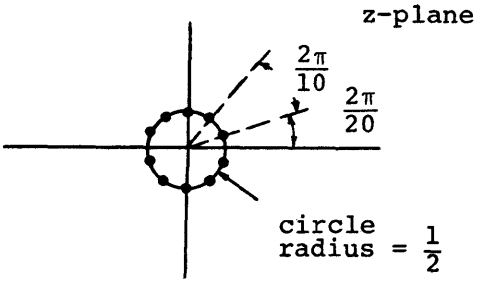


Figure P9.6-2

Problem 9.7*

Consider a finite-duration sequence x(n), which is zero for n < 0 and n ≥ N, where N is even. Let the z-transform of x(n) be denoted by X(z). Listed here are two tables. In Table P9.7-1 are seven sequences obtained from x(n). In Table P9.7-2 are nine sequences obtained from X(z). For each sequence in Table P9.7-1, find its DFT in Table P9.7-2. The size of the transform considered must be greater than or equal to the length of the sequence $g_k(n)$. For purposes of illustration only assume that x(n) can be represented by the envelope shown in figure P9.7-1

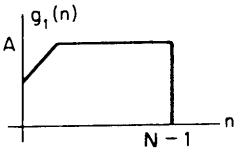
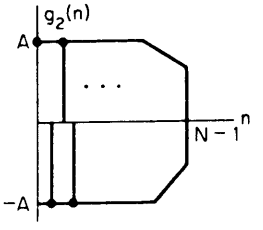
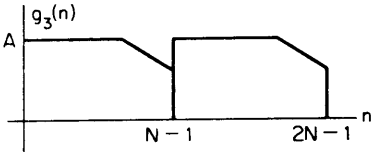
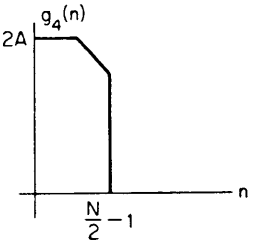
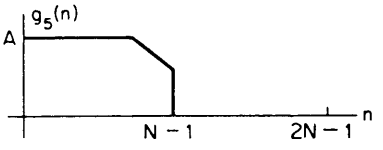
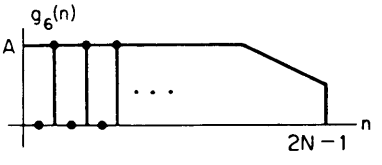
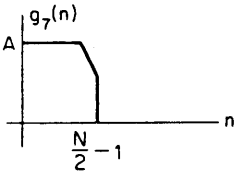
$g_1(n) = x(N-1-n)$	
$g_2(n) = (-1)^n x(n)$	
$g_3(n) = \begin{cases} x(n), & 0 \le n \le N-1 \\ x(n-N), & N \le n \le 2N-1 \\ 0, & \text{otherwise} \end{cases}$	
$g_4(n) = \begin{cases} x(n) + x(n+N/2), & 0 \le n \le N/2-1 \\ 0, & \text{otherwise} \end{cases}$	
$g_5(n) = \begin{cases} x(n), & 0 \le n \le N-1 \\ 0, & N \le n \le 2N-1 \\ 0, & \text{otherwise} \end{cases}$	
$g_6(n) = \begin{cases} x\left(\frac{n}{2}\right), & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$	
$g_7(n) = x(2n)$	

Table P9.7-1

$H_1(k) = X(e^{j2\pi k/N})$

$H_2(k) = X(e^{j2\pi k/2N})$

$$H_3(k) = \begin{cases} 2X(e^{j2\pi k/2N}), & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$$

$H_4(k) = X(e^{j2\pi k/(2N-1)})$

$H_5(k) = 0.5[X(e^{j2\pi k/N}) + X(e^{j2\pi(k+N/2)/N})]$

$H_6(k) = X(e^{j4\pi k/N})$

$H_7(k) = e^{j2\pi k/N} X(e^{-j2\pi k/N})$

$H_8(k) = X(e^{j(2\pi/N)(k+N/2)})$

$H_9(k) = X(e^{-j2\pi k/N})$

Table P9.7-2

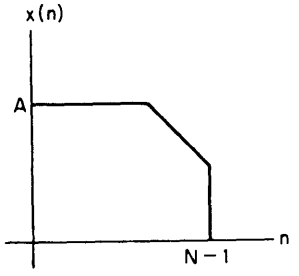


Figure P9.7-1

Solution 9.1

(a)

$$X(k) = \left[\sum_{n=0}^{N-1} x(n)\ W_N^{kn}\right] R_N(k)$$

$$= R_N(k)$$

(b)

$$X(k) = W_N^{kn_0}\ R_N(k)$$

(c)

$$X(k) = \left[\sum_{n=0}^{N-1} a^n\ W_N^{kn}\right] R_N(k)$$

$$= \left[\frac{1-a^N\ W_N^{kN}}{1-a\ W_N^{k}}\right] R_N(k)$$

$$= \left[\frac{1-a^N}{1-a\ W_N^{k}}\right] R_N(k)$$

Solution 9.2

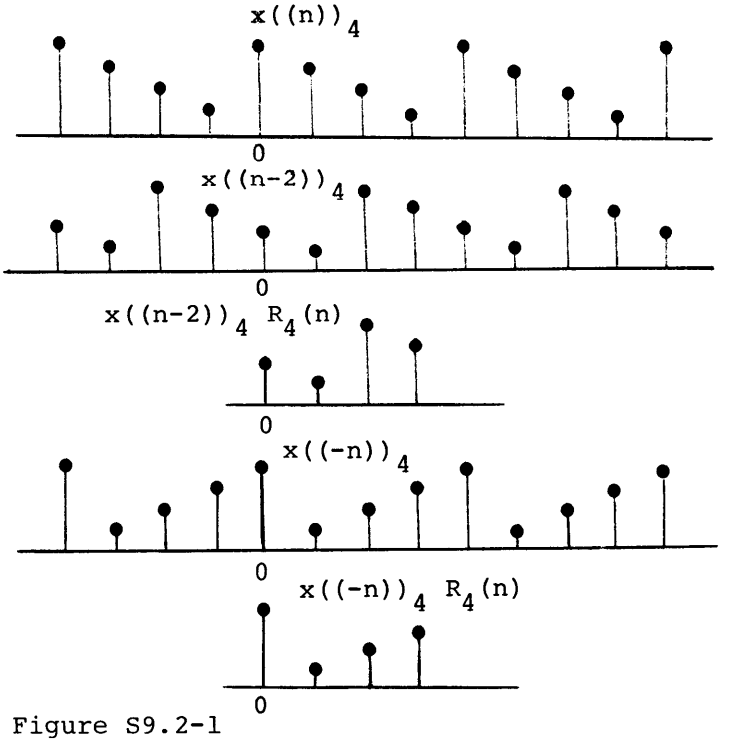


Figure S9.2-1

Solution 9.3

Since $X_I(k) = - X_I((N - k))_N \, R_N(k)$

$X_I(0) = - X_I((N))_N \, R_N(k) = - X_I(0)$

Therefore $X_I(0) = 0$

Also, for N even,

$X_I(\frac{N}{2}) = - X_I((N - \frac{N}{2}))_N \, R_N(k)$

$= - X_I(\frac{N}{2})$

Therefore $X_I(\frac{N}{2}) = 0.$

Solution 9.4

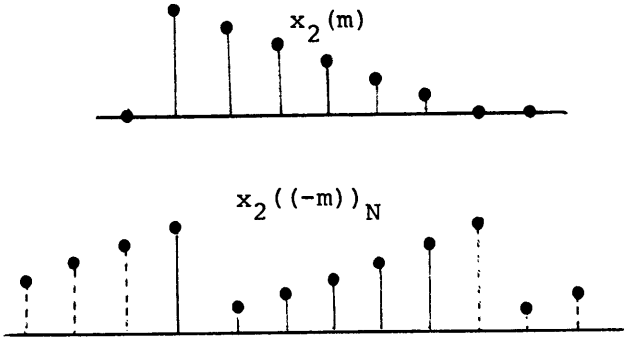


Figure S9.4-1

Solution 9.5

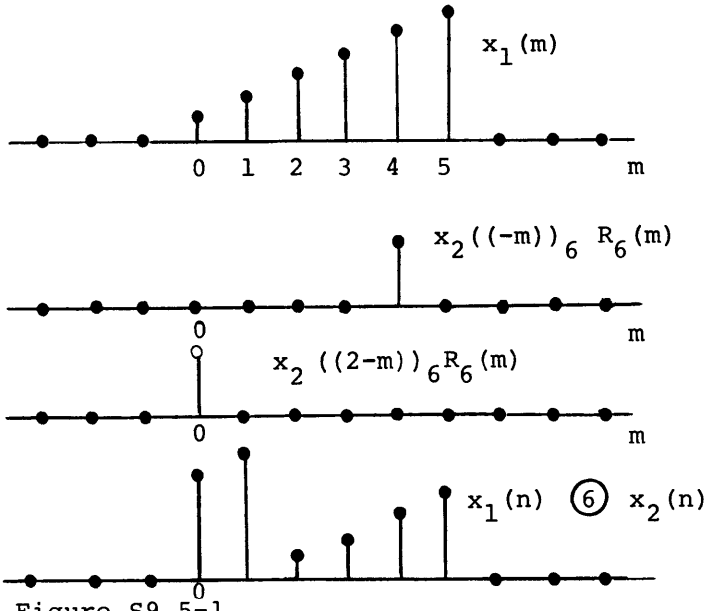


Figure S9.5-1

Note that this corresponds to $x_1(n)$ circularly shifted to the right by two points.

Solution 9.6

We wish to compute $X_1(k)$ given by

$$X_1(k) = \left\{ \sum_{n=0}^{9} x(n)\ z_k^{-n} \right\} R_{10}(k)$$

where $z_k = 0.5\ e^{j\frac{2\pi k}{10}}\ e^{j\frac{\pi}{10}}$

so that

$$X_1(k) = \left\{ \sum_{n=0}^{9} x(n) \left[\frac{1}{2}\ e^{j\frac{2\pi k}{10}}\ e^{\frac{j\pi}{10}} \right]^{-n} \right\} R_{10}(k)$$

$$= \left\{ \sum_{n=0}^{9} x(n) \left[\frac{1}{2}\ e^{\frac{j\pi}{10}} \right]^{-n}\ e^{-j\frac{2\pi kn}{10}} \right\} R_{10}(k)$$

Thus $X_1(k)$ is the 10-point DFT of the sequence

$$x_1(n) = x(n) \left[\frac{1}{2}\ e^{\frac{j\pi}{10}} \right]^{-n}$$

Solution 9.7

In all of the following equations the DFT computed is valid only in the range $0 \le k \le N-1$ and is zero outside that range. This permits us to keep the equations somewhat cleaner by suppressing the use of the function $R_N(k)$.

$$G_1(k) = \sum_{n=0}^{N-1} x(N-1-n)\ W_N^{kn}$$

$$= \sum_{m=0}^{N-1} x(m)\ W_N^{k(N-1-m)}$$

$$= \sum_{m=0}^{N-1} x(m)\ e^{j\frac{2\pi k}{N}}\ e^{j\frac{2\pi}{N}mk}$$

$$= e^{j\frac{2\pi k}{N}}\ X(e^{-j\frac{2\pi}{N}k}) = H_7(k)$$

$$G_2(k) = \sum_{n=0}^{N-1} (-1)^n\ x(n)\ W_N^{kn} = \sum_{n=0}^{N-1} x(n)\ W_N^{\frac{Nn}{2}}\ W_N^{kn}$$

$$= \sum_{n=0}^{N-1} x(n)\ e^{-j\frac{2\pi}{N}(k+\frac{N}{2})n} = X(e^{j\frac{2\pi}{N}(k+\frac{N}{2})})$$

$$= H_8(k)$$

$$G_3(k) = \sum_{n=0}^{N-1} x(n)\ W_{2N}^{nk} + \sum_{n=N}^{2N-1} x(n-N)\ W_{2N}^{nk}$$

$$= \sum_{n=0}^{N-1} x(n) \left[W_{2N}^{nk} + W_{2N}^{(n+N)k} \right]$$

$$= \sum_{n=0}^{N-1} x(n)\ W_{2N}^{nk} \left[1 + W_{2N}^{Nk} \right]$$

$$= \left[1 + (-1)^k \right] X(e^{j\frac{2\pi}{2N}k}) = H_3(k)$$

$$G_4(k) = \sum_{n=0}^{\frac{N}{2}-1} \left(x(n) + x(n+\tfrac{N}{2}) \right) W_{\frac{N}{2}}^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n)\ W_{\frac{N}{2}}^{nk} + \sum_{n=\frac{N}{2}}^{N-1} x(n)\ W_{\frac{N}{2}}^{(n-\frac{N}{2})k}$$

$$= \sum_{n=0}^{N-1} x(n)\ W_{\frac{N}{2}}^{nk} = X(e^{j\frac{4\pi}{N}}) = H_6(k)$$

$$G_5(k) = \sum_{n=0}^{2N-1} x(n)\ W_{2N}^{nk} = X(e^{j\frac{\pi k}{N}}) = H_2(k)$$

$$G_6(k) = \sum_{n=0}^{N-1} x(n)\ W_{2N}^{2nk} = X(e^{j\frac{2\pi k}{N}}) = H_1(k)$$

$$G_7(k) = \sum_{n=0}^{\frac{N}{2}-1} x(2n)\ W_{\frac{N}{2}}^{nk}$$

$$= \sum_{n=0}^{N-1} x(n) \left[\frac{1+(-1)^n}{2} \right] W_{\frac{N}{2}}^{nk/2}$$

$$= \frac{1}{2} \sum_{n=0}^{N-1} x(n) \left[W_N^{nk} + W_N^{n(k+N/2)} \right]$$

$$= \frac{1}{2}\left[X\left(e^{j\frac{2\pi k}{N}}\right) + X\left(e^{j\frac{2\pi}{N}(k+N/2)}\right)\right] = H_5(k)$$

All of the above properties can alternatively be obtained from the basic DFT properties of sections 8.7 and 8.8, or the z-transform properties of section 4.4. Many of the properties used in this problem have important practical applications. $g_5(n)$, for example, corresponds to augmenting a finite length sequence with zeros so that a computation of the DFT for this augmented sequence provides finer spectral sampling of the Fourier transform.

CIRCULAR CONVOLUTION

1. Lecture 10 - 43 minutes

Circular Convolution $x_3(n) = x_1(n) \textcircled{N} x_2(n)$

$$= \left[\sum_{m=0}^{N-1} \tilde{x}_1(m)\, \tilde{x}_2(n-m)\right] \mathcal{R}_N(n)$$

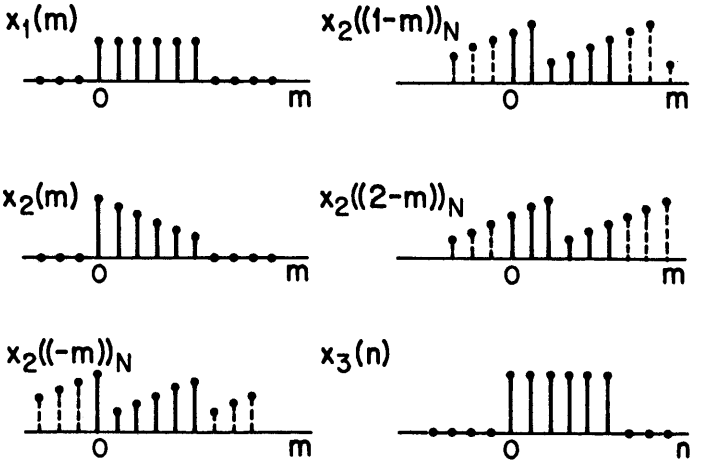
$$= \left[\sum_{m=0}^{N-1} x_1((m))_N\, x_2((n-m))_N\right] \mathcal{R}_N(n)$$

$$= \left[\sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N\right] \mathcal{R}_N(n)$$

$$= \left[x_1(n) * x_2((n))_N\right] \mathcal{R}_N(n)$$

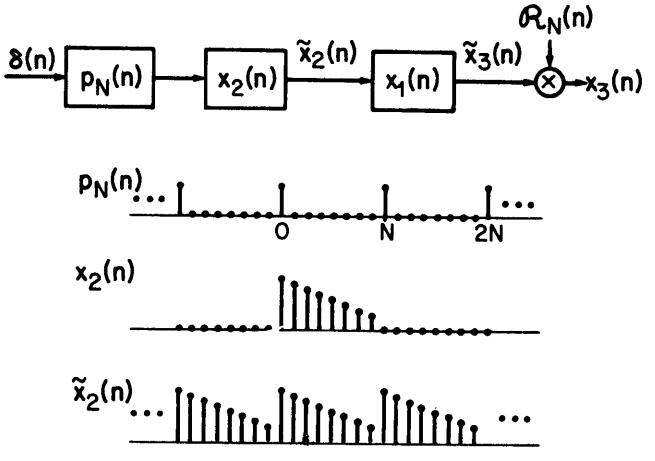
Circular convolution expressed in terms of periodic and linear convolution.

a.



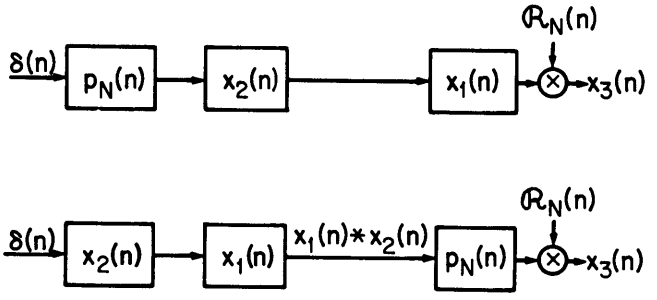
Example of circular convolution of two sequences.

b.



An interpretation of circular convolution.

c.



Rearrangement of the operations in forming the circular convolution.

d.

"Circular Convolution =
Linear Convolution + Aliasing"

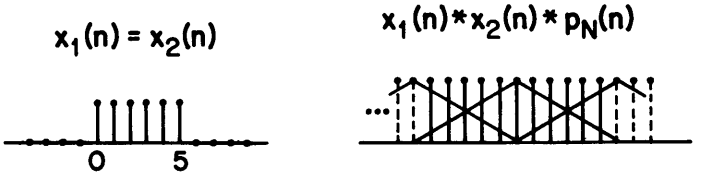
$$\hat{x}_3(n) = x_1(n) * x_2(n)$$

$$x_3(n) = x_1(n) \textcircled{N} x_2(n)$$

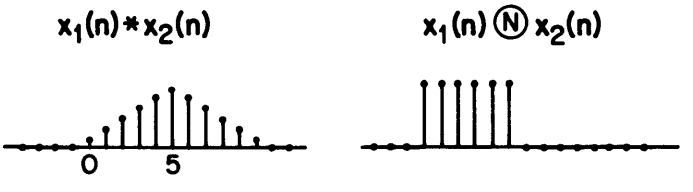
$$x_3(n) = \left[\sum_{r=-\infty}^{+\infty} \hat{x}_3(n+rN)\right]\mathcal{R}_N(n)$$

Interpretation of circular convolution as linear convolution followed by aliasing.

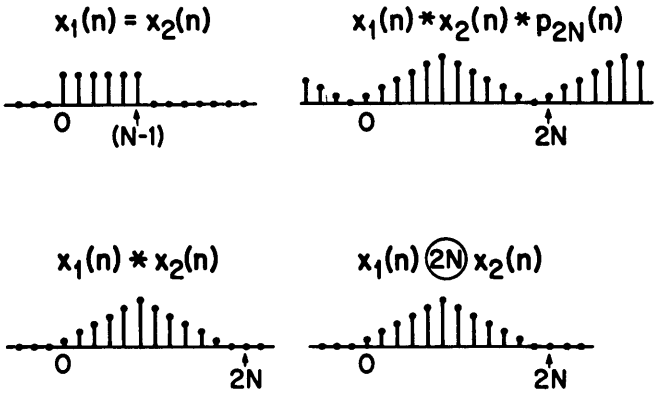
e.



Example of a circular convolution formed by linear convolution followed by aliasing.

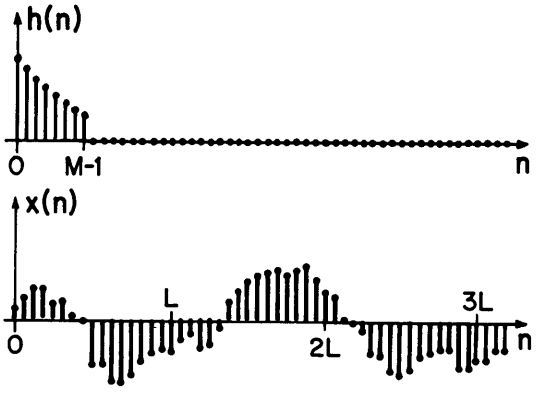


f.



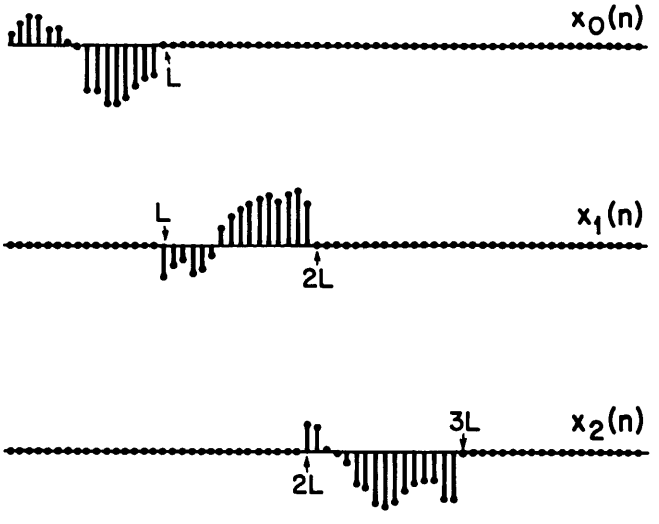
Obtaining a linear convolution through the use of circular convolution.

g.



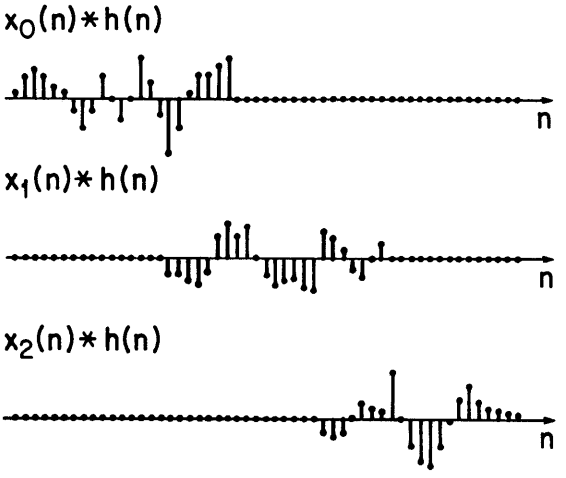
A finite length unit sample response and a sequence of indefinite length.

h.



Sectioning of the sequence x(n).

i.



Linear convolution of h(n) with the sections of x(n). Note the overlap in the resulting output sections.

j.

2. Comments

In the last lecture we introduced the property of circular convolution for the Discrete Fourier Transform. The fact that multiplication of DFT's corresponds to a circular convolution rather than a linear convolution of the original sequences stems essentially from the implied periodicity in the use of the DFT, i.e. the fact that it essentially corresponds to the Discrete Fourier series of a periodic sequence. In this lecture we focus entirely on the properties of circular convolution and its relation to linear convolution. An interpretation of circular convolution as linear convolution followed by aliasing is developed.

As we will see in a later lecture, there is a highly efficient algorithm for the computation of the DFT and consequently it is often useful in practice to implement a convolution (for implementing a filter, for example) by computing the Discrete Fourier Transforms, multiplying, and then inverse transforming. This leads, of course, to a circular convolution, whereas it is generally a linear convolution that is desired. Consequently an important consideration is the use of circular convolution to implement a linear convolution. This can be accomplished by extending the lengths of the sequences to be convolved by "padding" with zeros. An additional consideration is the fact that a signal to be filtered may be arbitrarily long. Filtering can still be accomplished in this case using the DFT by first sectioning the input into finite length segments and implementing the convolution for each of these. The filtered segments can then be added with the appropriate synchronization

to produce the desired output. There are two basic procedures which are commonly used, referred to as the "overlap-add" and "overlap-save" methods. Although only the first of these is presented in the lecture, both are discussed in the text.

3. Reading

Text: Sections 8.7.5. (page 542) and 8.9.

4. Problems

Problem 10.1

Let x(n) and h(n) denote two finite length sequences, both of length N.

(a) What is the maximum possible length of the linear convolution of x(n) with h(n)?

(b) What is the maximum possible length of the N-point circular convolution of x(n) with h(n)?

Problem 10.2

In the figure below is shown a four-point sequence x(n).

(a) Sketch the linear convolution of x(n) with x(n).

(b) Sketch the four point circular convolution of x(n) with x(n).

(c) Sketch the ten-point circular convolution of x(n) with x(n).

(d) What is the smallest value of N for which an N-point circular convolution of x(n) with x(n) will be identical to the linear convolution?

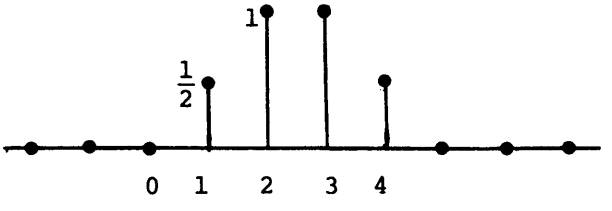


Figure P10.2-1

Problem 10.3

Consider two finite-duration sequences x(n) and y(n) where both are zero for n < 0 and with

$x(n) = 0, \quad n \geq 8$

$y(n) = 0, \quad n \geq 20$

The 20-point DFTs of each of the sequences are multiplied and the inverse DFT computed. Let r(n) denote the inverse DFT. Specify which points in r(n) correspond to points that would be obtained in a linear convolution of x(n) and y(n).

Problem 10.4*

We want to filter a very long string of data with a FIR filter whose unit-sample response is 50 samples long. We wish to implement this filter with an FFT using the overlap-save technique. To do this: (1) The input sections must be overlapped by V samples; and (2) from the output due to each section we must extract M samples such that when these samples from each section are butted together, the resulting sequence is the desired filtered output. Assume that the input segments are 100 samples long and that the size of the DFT is 128 ($=2^7$) points. Further assume that the output sequence from the circular convolution is indexed from point 0 to point 127.

(a) Determine V.

(b) Determine M.

(c) Determine the index of the beginning and the end of the M points extracted; i.e., determine which of the 128 points from the circular convolution is extracted to be abutted with the result from the previous section.

Problem 10.5*

The problem often arises in which a signal x(n) has been filtered by a linear time-invariant system that results in a distorted signal y(n) and we wish to recover the original signal. This can often be done by processing y(n) with a linear time-invariant system whose impulse response is such that the overall impulse response of the two systems in cascade is a unit sample. This is generally referred to as inverse filtering.

We have discussed the procedure for implementing an FIR filter using the DFT. The procedure involves, in part, multiplying the DFT of the input, X(k) (or input sections), by H(k), the DFT of the system unit sample response to produce Y(k), the DFT of the output. It is often suggested, incorrectly, that the sequence $h_1(n)$ whose DFT is 1/H(k), is the impulse response of the inverse filter. The purpose of this problem is to indicate why that suggestion is incorrect.

Consider a linear time-invariant system with impulse response h(n) given by

$$h(n) = \delta(n) - \frac{1}{2}\delta(n - n_0)$$

This system is an idealized example of a system that would introduce reverberation. Assume that $N = 4\ n_0$.

(a) Determine the N-point DFT H(k) of h(n).

(b) Now consider the N-point DFT $H_1(k)$ of a sequence $h_1(n)$ specified by

$$H_1(k) = \frac{1}{H(k)} \quad k = 0,1,\ldots, N - 1$$

Determine $h_1(n)$. Hint: If you have trouble evaluating the IDFT summation directly, express $H_1(k)$ as a polynomial in $W_N^{n_0 k}$ and observe that the $h_1(n)$ are coefficients of W_N^{nk}.

(c) By evaluating the linear convolution of h(n) and $h_1(n)$, show that $h(n) * h_1(n)$ is not a unit sample $\delta(n)$ and, consequently, $h_1(n)$ is not the unit sample response for the inverse system.

(d) Compute the N-point circular convolution of h(n) and $h_1(n)$.

(e) Determine the unit-sample response $h_i(n)$ of the inverse sytem for h(n). This can be done in a number of ways. One is to note that with H(z) and $H_i(z)$ denoting the z-transforms of h(n) and $h_i(n)$, $H_i(z) = 1/H(z)$. The inverse z-transforms of $H_i(z)$ can then be evaluated by long division.

(f) Determine and verify numerically the relationship between $h_1(n)$ and $h_i(n)$.

CIRCULAR CONVOLUTION

Solution 10.1

(a) It is straightforward to see graphically that the maximum possible length of the linear convolution is 2N-1. Alternatively,

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k)\ h(n - k)$$

$x(k)$ is zero outside the range $0 \leq k \leq N - 1$ and $h(n - k)$ is zero outside the range $-N + 1 + n \leq k \leq n$. These two intervals overlap only for $0 \leq n \leq 2N - 2$, and consequently $y(n)$ must be zero outside that range.

(b) The circular convolution of two sequences of length N has a maximum length of N. This can be seen in a number of ways. For example, the circular convolution corresponds to extracting one period of the periodic convolution of $x(n)$ and $h(n)$. Since the periodic convolution is periodic with period N, each period is of maximum length N.

Solution 10.2

(a)

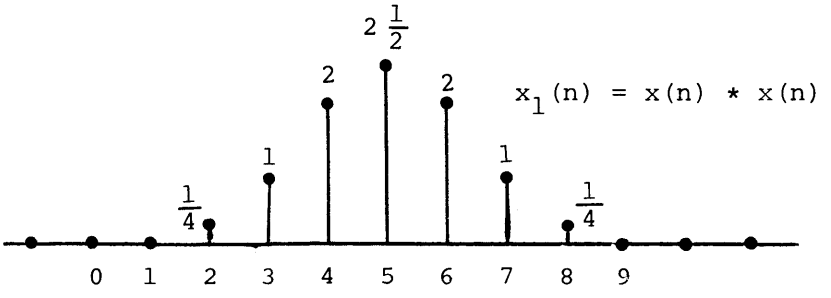


Figure S10.2-1

(b) We can obtain the four-point circular convolution by "aliasing" the linear convolution. Thus

$$x(n) \textcircled{4} x(n) = \left[\sum_{r=-\infty}^{+\infty} x_1(n + 4r)\right] R_4(n)$$

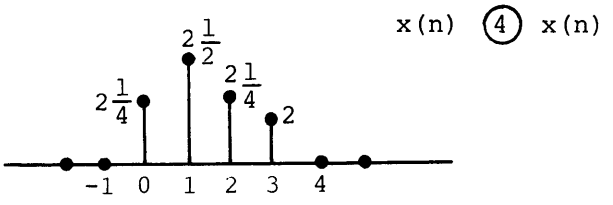


Figure S10.2-2

(c) The ten-point circular convolution can be obtained in the same way. In this case, however, since $x_1(n)$ is of length 7, the delayed replicas of $x_1(n)$ in the "aliasing" equation do not overlap. Thus $x(n) \textcircled{10} x(n)$ is identical to the linear convolution as obtained in (a).

(d) The circular convolution will be identical to the linear convolution if the delayed replicas of the linear convolution have no non-zero values overlapping. For this specific example that will be the case for $N \geq 9$. More generally, from problem 10.1 (a) we know that the linear convolution of an N_1 point sequence with itself will have a maximum length $(2N - 1)$ and consequently the $(2N - 1)$ point circular convolution of an N-point sequence with itself will be identical to the N-point linear convolution.

Solution 10.3

This is most easily done by again considering circular convolution as "linear convolution plus aliasing." In the figure below we indicate the linear convolution of x(n) and y(n). Since x(n) is of length 8 and y(n) is of length 20, the linear convolution, which we'll denote by w(n) is of length 27.

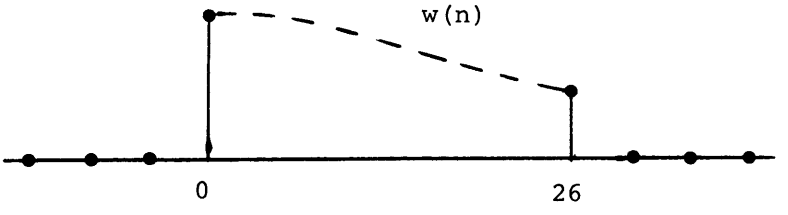


Figure S10.3-1

The 20-point circular convolution can be obtained by adding w(n) delayed by integer multiples of 20, and then extracting the first 20 points.

From Figure S10.3-2 we observe that in the interval $0 \leq n \leq 19$ there is aliasing only in the first seven points. The remaining thirteen points, i.e. for $8 \leq n < 20$ the points in the linear convolution remain undisturbed. Thus it is these points in r(n) that correspond to points that would be obtained in a linear convolution of x(n) and y(n).

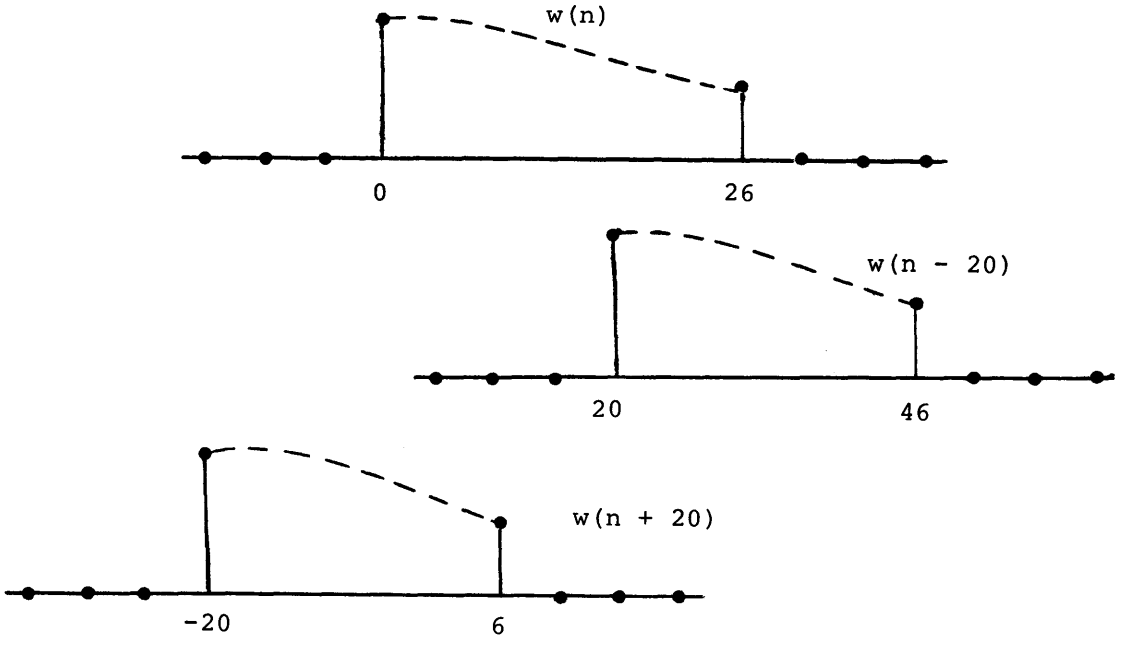


Figure S10.3-2

Solution 10.4*

The linear convolution of the unit-sample response with an input section is of length 149. In the 128-point circular convolution the first 49 points are "aliased" and the remaining 79 points correspond to a linear convolution of the unit-sample response and the input section. However because the length of the DFT was greater than the length of the input section, the section was augmented with 28 zeros. Thus the last 28 points could not simply be abutted with the results from previous sections. Hence in the circular convolution the first 49 points and the last 28 points must be discarded. The input sections are then overlapped by 49 points and the 51 points indexed from 49 through 99 are abutted with the corresponding points from the preceding section.

Solution 10.5*

(a) $H(k) = 1 - \frac{1}{2} W_N^{kn_0}$

(b) $H_1(k) = \frac{1}{1-\frac{1}{2} W_N^{kn_0}} = \sum_{\ell=0}^{\infty} (\frac{1}{2})^{\ell} W_N^{\ell kn_0}$

letting $\ell = n + 4r$, $H_1(k)$ can be rewriten as

$$H_1(k) = \sum_{n=0}^{3} \sum_{r=0}^{\infty} (\frac{1}{2})^{n+4r} W_N^{kn_0(n+4r)}$$

or, since

$N = 4n_0$, $W_N^{kn_0 4r} = 1$ and

$$H_1(k) = \sum_{n=0}^{3} W_N^{kn_0 n} \sum_{r=0}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{4r}$$

$$= \frac{16}{15} \sum_{n=0}^{3} \left(\frac{1}{2}\right)^n W_N^{kn_0 n}$$

Therefore $h_1(n) = \frac{16}{15}\left[\delta(n) + \frac{1}{2}\,\delta(n - n_0) + \frac{1}{4}\,\delta(n - 2n_0) + \frac{1}{8}\,\delta(n - 3n_0)\right]$

(c) and (d) It is straightforward to see that the linear convolution of $h(n)$ with $h_1(n)$ is not a unit-sample and hence $h_1(n)$ is not the unit-sample response of the inverse system. The <u>circular</u> convolution of $h(n)$ with $h_1(n)$ is however a unit-sample since $H_1(k) = \frac{1}{H(k)}$

(e) $H(z) = 1 - \frac{1}{2}\,z^{-n_0}$

$$H_i(z) = \frac{1}{1-\frac{1}{2}\,z^{-n_0}} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n\,n_0}$$

Therefore $h_i(n) = \sum_{\ell=0}^{+\infty} \left(\frac{1}{2}\right)^{\ell}\,\delta(n - \ell n_0)$

(f) $h_1(n)$ as determined in part (b) is an aliased version of $h_i(n)$, i.e.

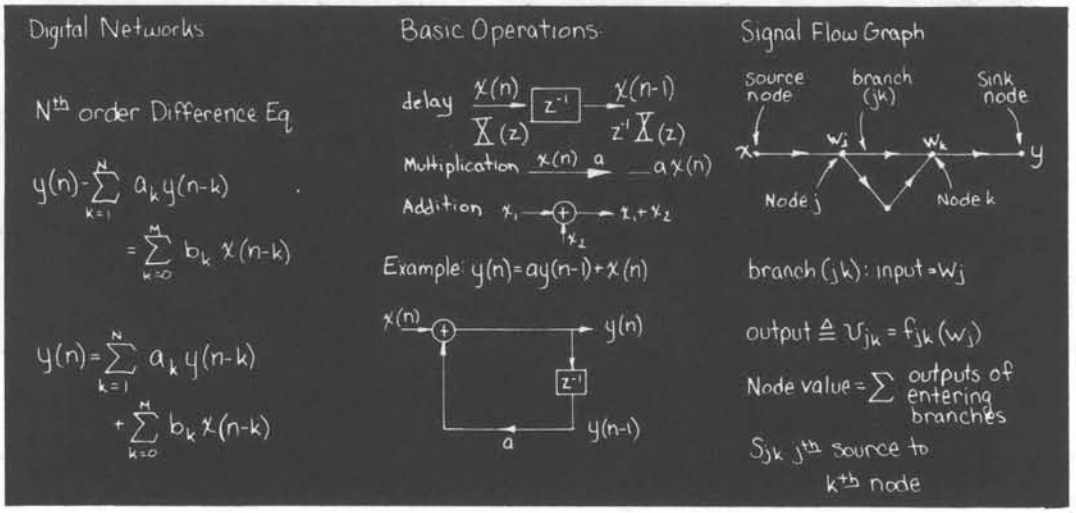
$$h_1(n) = \left[\sum_{r=-\infty}^{+\infty} h_i(n - rN)\right] R_N(n)$$

This can be interpreted in terms of the fact that $H_1(k)$ corresponds to sampling $H_i(z)$ on the unit circle. This sampling in the frequency domain corresponds to aliasing in the time domain.

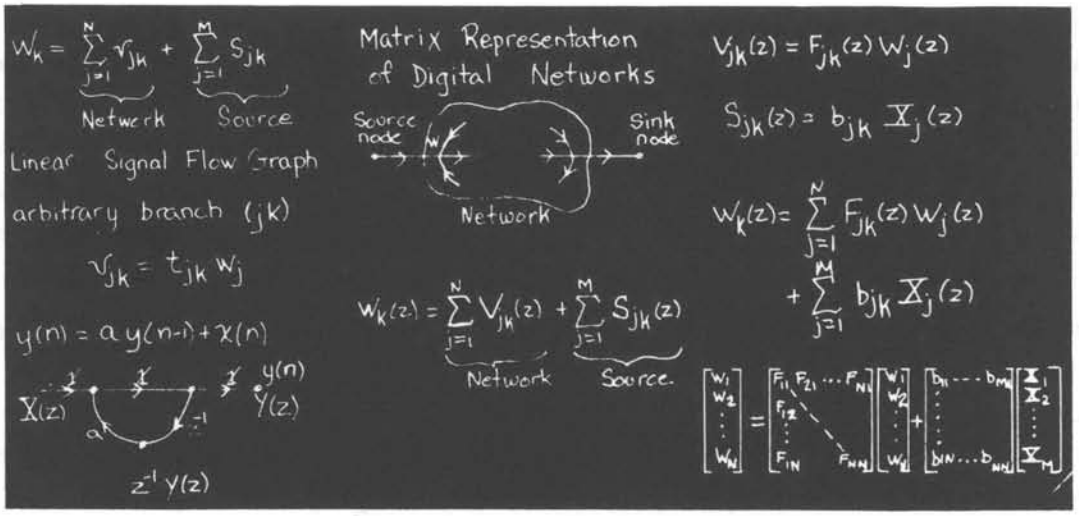
REPRESENTATION OF LINEAR DIGITAL NETWORKS

1. Lecture 11 - 52 minutes

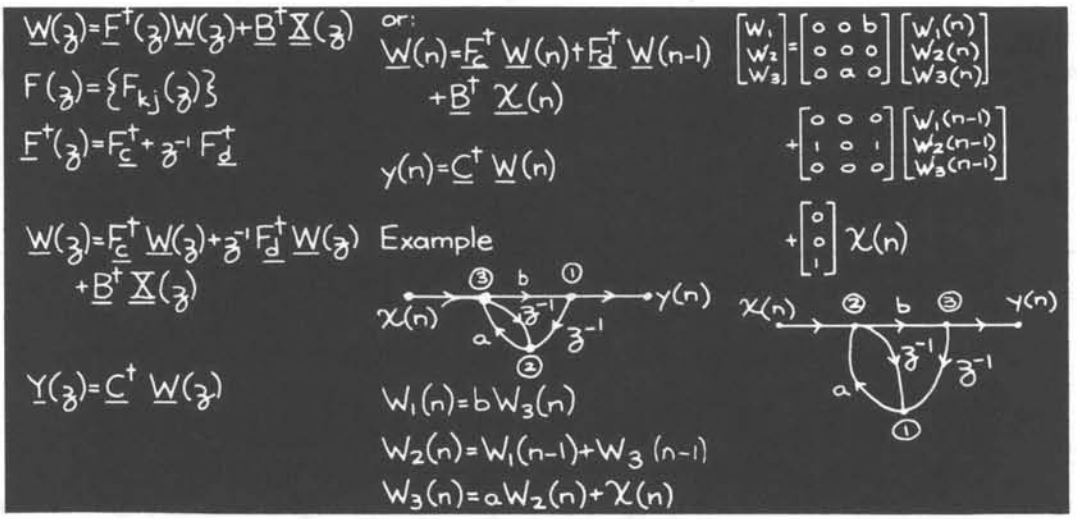
a.



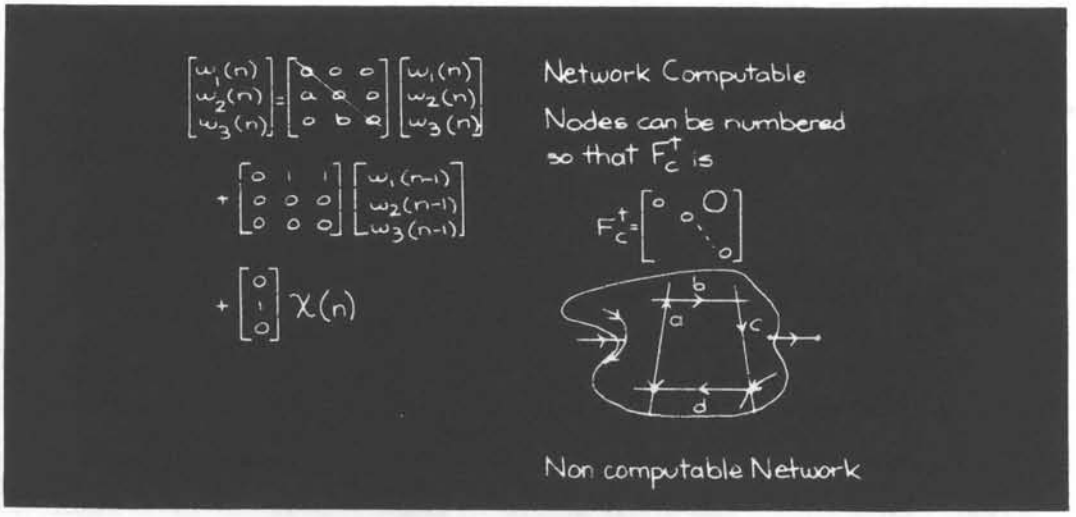
b.



c.



d.


Network Computable
Nodes can be numbered so that F_c^t is
$F_c^t = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$
a
b
c
d
Non computable Network

2. Comments

The class of discrete-time systems characterized by linear constant-coefficient difference equations plays a particularly important role in digital signal processing, in part because many of the issues involved in their implementation are well understood. In this lecture we discuss the representation of this class of systems in terms of signal-flow graphs and matrices. In the two following lectures we will utilize this representation to discuss a number of basic network structures for infinite impulse response and finite impulse response digital filters.

This lecture is directed primarily toward introducing the notation of signal-flow graphs and the corresponding matrix representation. Although the concept of computability is discussed briefly in the lecture we will not be relying on this idea in the following lectures. The notion of computability emphasizes, however, the fact that we are viewing the signal-flow graph as representing a computational algorithm for the system, i.e. it represents an implementation of a digital filter. Some flow-graph representations of a digital filter may be mathematically valid but cannot be implemented. This class corresponds to signal-flow graphs which contain delay-free loops.

3. Reading

Text: Sections 6.0 (page 290) through 6.2.

4. Problems

Problem 11.1

Consider the discrete-time system represented by the linear constant-

coefficient difference equation

$$y(n) - \frac{3}{4}\, y(n-1) + \frac{1}{8}\, y(n-2) = x(n)$$

(a) Draw a block-diagram representation of the system in terms of adders and delay and coefficient multiplication branches.

(b) Draw a linear signal-flow graph representation of the system.

Problem 11.2

In figure P11.2-1 is shown a digital network.

(a) For this network, determine the matrices $\underline{F}^t$, $\underline{B}^t$ and $\underline{C}^t$ as specified on chalkboard (c).

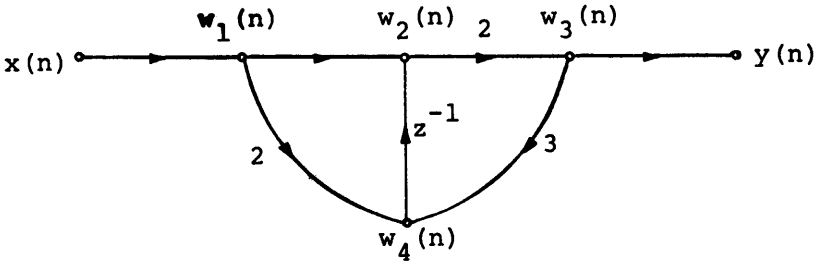


Figure P11.2-1

(b) Determine the matrices $\underline{F_c}^t$ and $\underline{F_d}^t$ in the alternative matrix representation on chalkboard (c).

Problem 11.3

Determine the system functions of the two networks in figure P11.3-1 and show that they have the same poles.

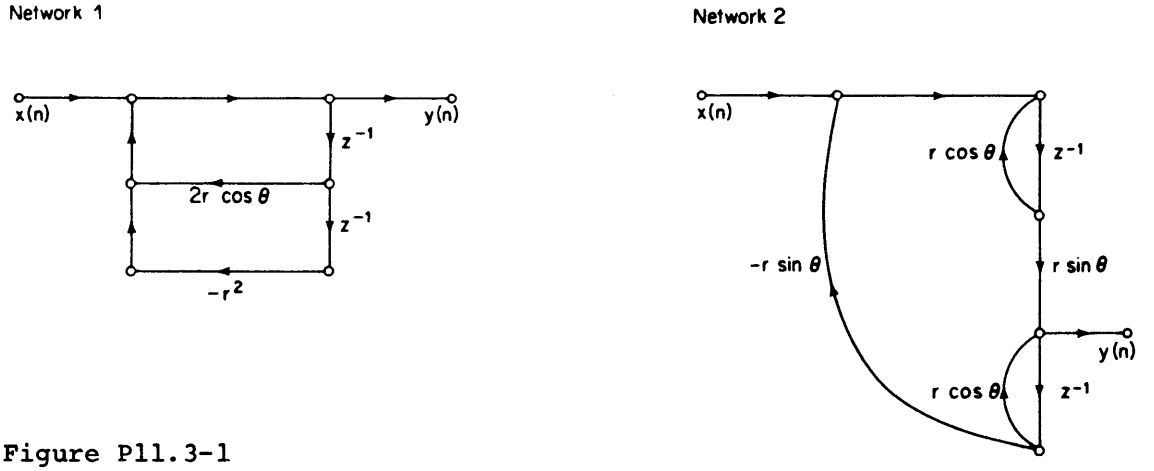


Figure P11.3-1

Problem 11.4

For the network shown in figure P11.4-1, determine all possible ordering of the nodes such that the node variables can be computed in sequence.

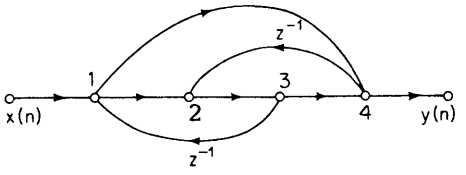


Figure P11.4-1

Problem 11.5*

In figure 11.5-1 is indicated a digital network A with two inputs, $x_1(n)$ and $x_2(n)$, and two outputs, $y_1(n)$ and $y_2(n)$.

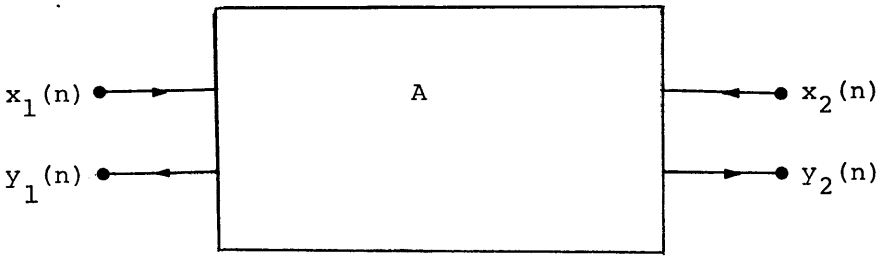


Figure P11.5-1

The network A can be described in terms of the two-part set of equations

$$Y_1(z) = H_1(z) + H_2(z)\ X_2(z)$$

$$Y_2(z) = H_3(z) + H_4(z)\ X_2(z)$$

where

$$H_1(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

$$H_2(z) = 1$$

$$H_3(z) = \frac{1 + 2z^{-1}}{1 + \frac{1}{2}z^{-1}}$$

$$H_4(z) = \frac{1}{1 + \frac{1}{2}z^{-1}}$$

(a) Draw a flow-graph implementation of the network. The transmittance of each branch must be a constant or a constant times z^{-1}. Higher-order functions of z^{-1} cannot be used as branch transmittances.

(b) We wish now to connect a network B to the network A as indicated in figure P11.5-2.

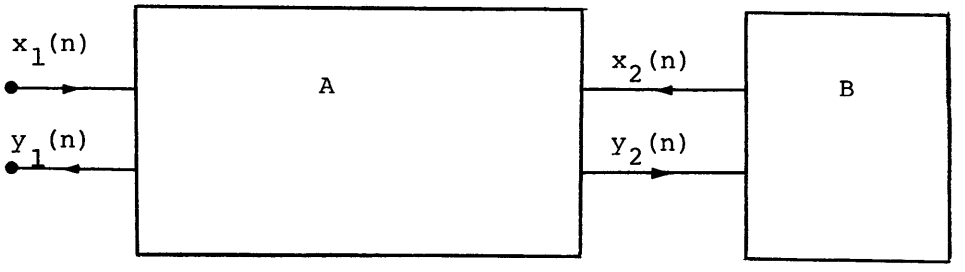


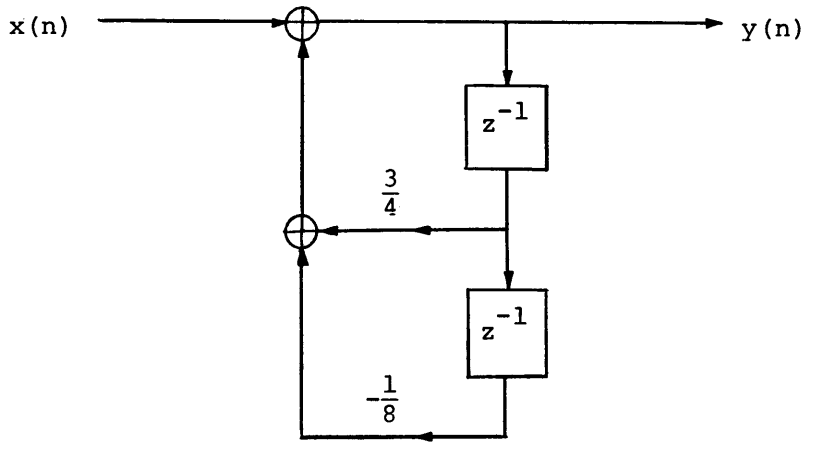
Figure P11.5-2

Network B has no internal source nodes and is characterized by the system function $X_2(z) = Y_2(z)\, H_B(z)$. Note that because of the notation used for network A, $y_2(n)$ denotes the input to network B and $x_2(n)$ is the output. By examining the flow graph in part (a) determine a necessary and sufficient condition on $H_B(z)$ so that the overall system is computable. If you have difficulty, first try the two possibilities $H_B(z) = 1$ and $H_B(z) = z^{-1}$.

REPRESENTATION OF LINEAR DIGITAL NETWORKS

Solution 11.1

(a)



(b)

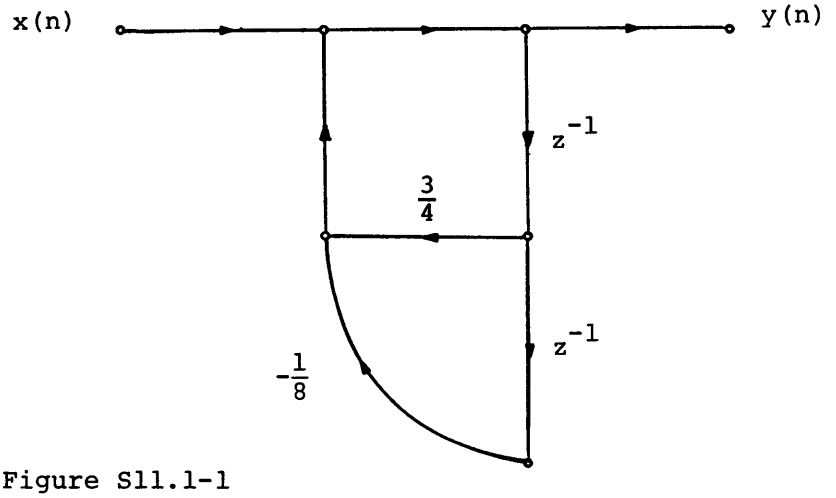


Figure S11.1-1

We have drawn the flow-graph to graphically correspond closely to the block diagram in (a). There are, of course, many other ways of drawing the flow-graph, for example

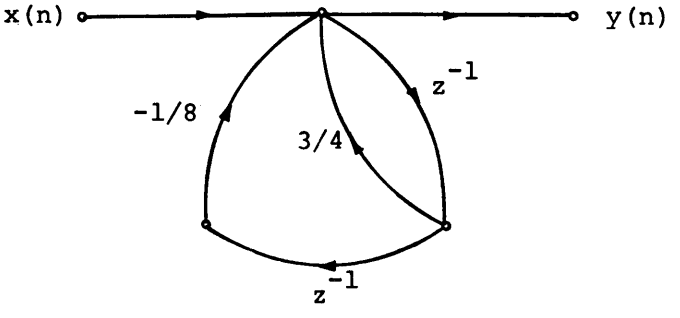


Figure S11.1-2

Solution 11.2

(a) The equations corresponding to this flowgraph are:

$$W_1(z) = X(z)$$

$$W_2(z) = W_1(z) + z^{-1} W_4(z)$$

$$W_3(z) = 2\, W_2(z)$$

$$W_4(z) = 2\, W_1(z) + 3\, W_3(z)$$

$$Y(z) = W_3(z)$$

or, in matrix form

$$\begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & z^{-1} \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} X(z)$$

$$Y(z) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix}$$

$$\underline{F}_c^{\,t} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \end{bmatrix}$$

and

$$\underline{F}_d^{\,t} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution 11.3

Let us carry this out by obtaining the transfer function for each of the networks. For network 1:

$$Y(z) = 2r \cos\theta\, z^{-1} Y(z) - r^2 z^{-2} Y(z) + X(z)$$

or

$$H_1(z) = 1/[1 - 2r \cos\theta\, z^{-1} + r^2 z^{-2}]$$

For network 2:
Define $W_1(z)$ as shown in Figure S11.3-1.

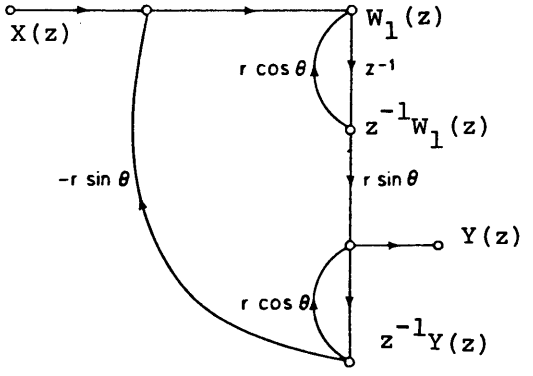


Figure S11.3-1

then

$$W_1(z) = X(z) - r \sin\theta\, z^{-1} Y(z) + r \cos\theta\, z^{-1} W_1(z)$$

$$Y(z) = r \sin\theta\, z^{-1} W_1(z) + r \cos\theta\, z^{-1} Y(z)$$

Solving for Y(z) in terms of X(z) we obtain

$$Y(z) = X(z)\, r(\sin\theta)\, z^{-1}/[1 - 2r \cos\theta\, z^{-1} + r^2 z^{-2}]$$

or

$$H_2(z) = r(\sin\theta)\, z^{-1}/[1 - 2r \cos\theta\, z^{-1} + r^2 z^{-2}]$$

Thus both networks have the same poles.

Solution 11.4

With the nodes ordered as shown in the figure, the matrix $\underline{F}_c^t$ is

$$\underline{F}_c^t = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Thus with the nodes arranged in the order 1-2-3-4 they can be computed in sequence since the matrix $\underline{F}_c^t$ is zero on and above the main diagonal. There is no other ordering possible.

Solution 11.5

(a) A flow-graph in terms of H_1, H_2, H_3 and H_4 can be drawn as

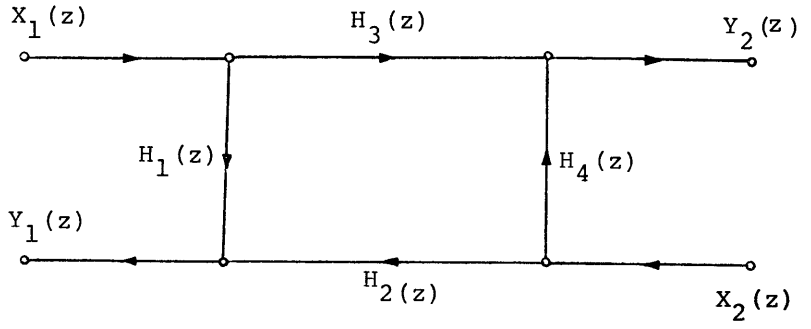


Figure S11.5-1

However we want to draw the flow-graph using branch transmittances which are constant or a constant times z^{-1}. Thus we replace H_1, H_2, H_3 and H_4 by their flow-graph implementations to obtain

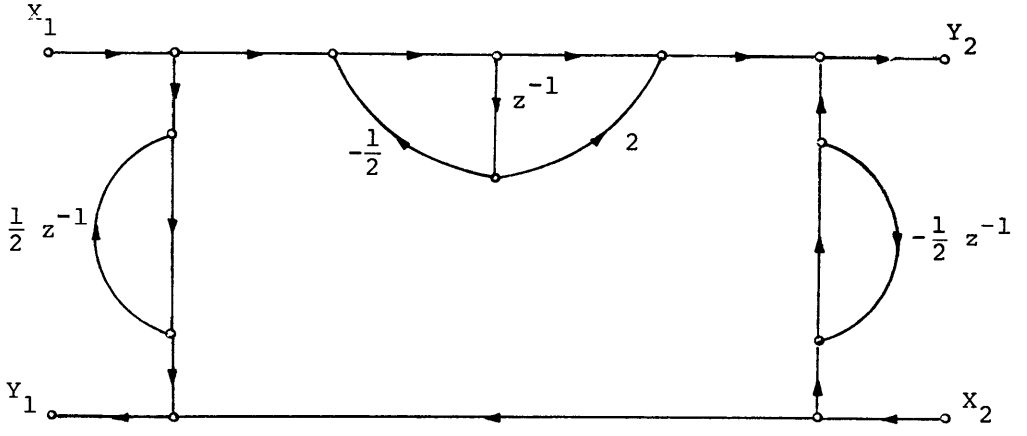


Figure S11.5-2

(b) We now want to connect a network $H_B(z)$ to the right-hand side of the above network. Observe that there is a delay-free path from X_2 to Y_2. Consequently, if the system $H_B(z)$ has a delay-free path from its input to its output, the total system will have a delay-free loop and thus will be noncomputable. By contrast, if $H_B(z)$ does not have a delay-free path from its input to its output, the overall system will be computable. A necessary and sufficient condition such that $H_B(z)$ is not delay-free from input to output is that $h_B(n)$, its unit sample

response be zero at n = 0 (we are, of course, assuming that the system is causal.) This is guaranteed if $H_B(z)$ can be written in the form

$$H_B(z) = z^{-1} \hat{H}_B(z)$$

where $\hat{H}_B(z)$ is also causal, or equivalently that

$$\lim_{z \to \infty} H_B(z) = 0.$$

NETWORK STRUCTURES FOR IIR SYSTEMS

1. Lesson 12 - 40 minutes

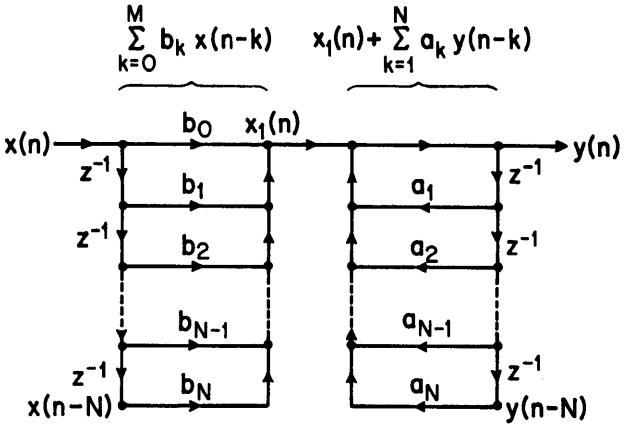
$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

$$y(n) = \sum_{k=0}^{M} b_k x(n-k) + \sum_{k=1}^{N} a_k y(n-k)$$

$$H(z) = \left[\sum_{k=0}^{M} b_k z^{-k}\right]\left[\frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}}\right]$$

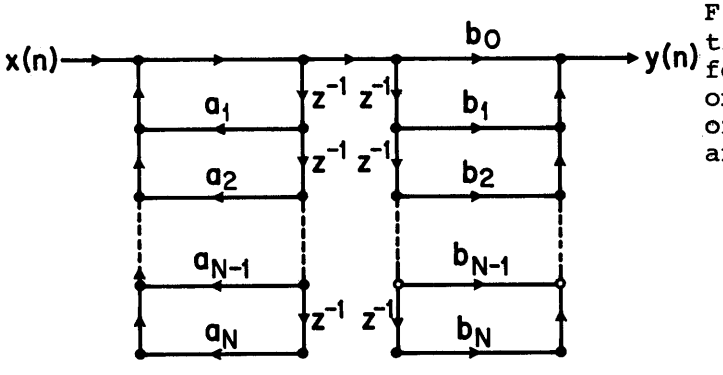
z-transform and difference equation for a general IIR system.

a.



Flow-graph representation of a general difference equation based on the factorization in b. (Direct form I realization.)

b.



Flow-graph representation of a general difference equation based on interchanging the order in which the poles and zeros are cascaded.

c.

$$y(n) = \sum_{k=0}^{M} b_k\, x(n-k) + \sum_{k=1}^{N} a_k\, y(n-k)$$

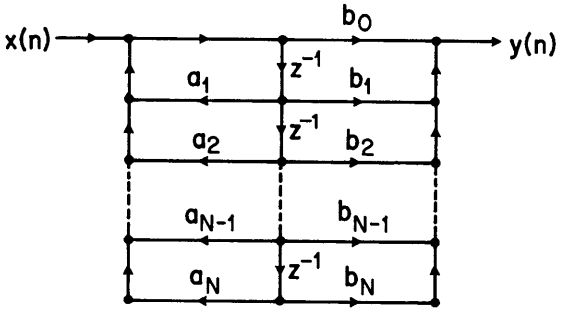
$$H(z) = \left[\frac{1}{1-\sum_{k=1}^{N} a_k z^{-k}}\right]\left[\sum_{k=0}^{M} b_k z^{-k}\right]$$

$$y_1(n) = x(n) + \sum_{k=1}^{N} a_k\, y_1(n-k)$$

$$y(n) = \sum_{k=0}^{M} b_k\, y_1(n-k)$$

z-tranform factorization and difference equation corresponding to the network in c.

d.



Flowgraph of c. collapsed to share delays (direct form II realization.)

e.

TRANSPOSITION THEOREM

1. REVERSE DIRECTION OF ALL BRANCHES
2. INTERCHANGE INPUT AND OUTPUT

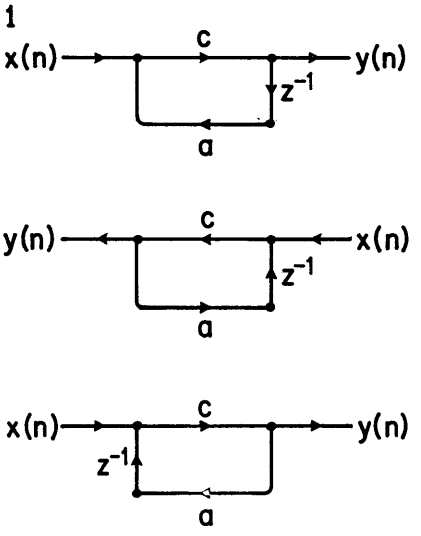
TRANSFER FUNCTION REMAINS THE SAME

Transposition theorem for signal flow-graphs.

f.

g.

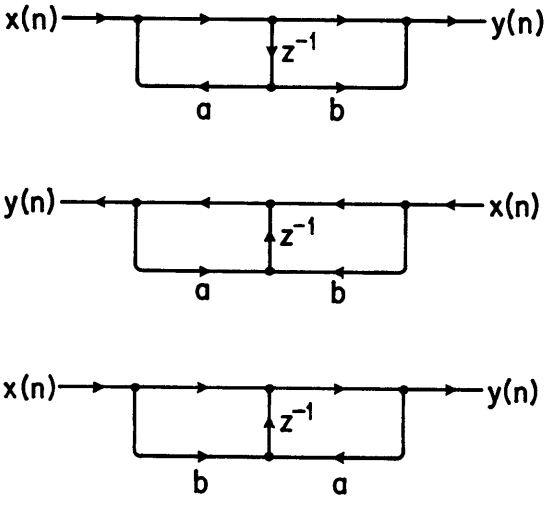
Example 1



Example of Transposition theorem.

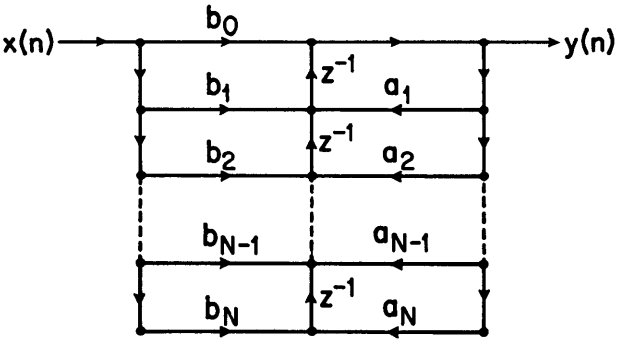
h.

Example 2



Example of Transposition theorem.

i.



Transposed direct form II structure.

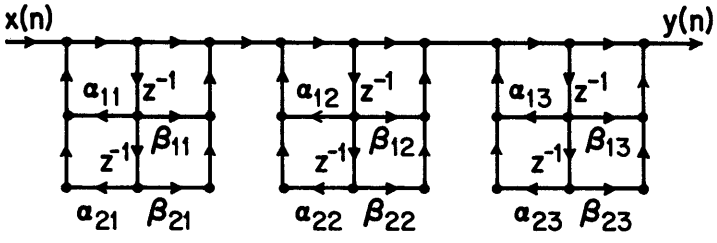
Cascade Structure

Factorization of the z-transform for the cascade structure.

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

$$= A\prod \frac{1+\beta_{1k} z^{-1} + \beta_{2k} z^{-2}}{1 - \alpha_{1k} z^{-1} - \alpha_{2k} z^{-2}}$$

j.



Cascade structure with a direct form II realization of each second-order subsystem.

k.

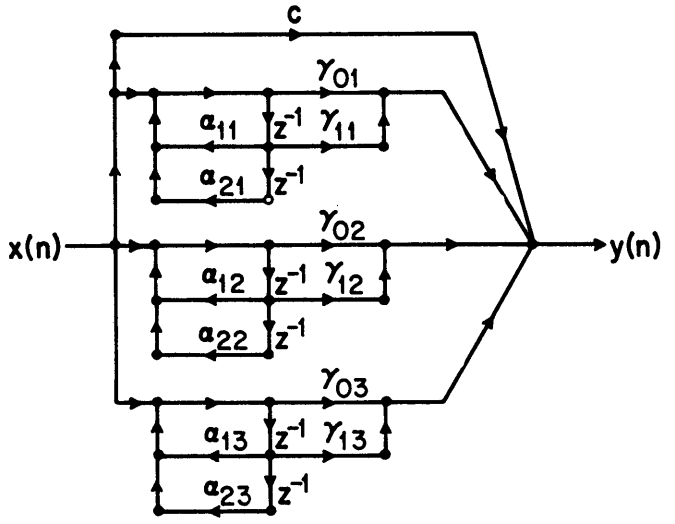
$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

Partial Fraction expansion for parallel structure.

$$= \sum_{k=1}^{N_1} \frac{A_k}{1 - c_k z^{-1}} + \sum_{k=1}^{N_2} \frac{B_k(1 - e_k z^{-1})}{(1 - d_k z^{-1})(1 - d_k^* z^{-1})} + \sum_{k=0}^{M-N} c_k z^{-k}$$

$$= \sum_{k=1}^{\frac{N+1}{2}} \frac{(\gamma_{0k} + \gamma_{1k} z^{-1})}{1 - \alpha_{1k} z^{-1} - \alpha_{2k} z^{-2}} + \sum_{k=0}^{M-N} c_k z^{-k}$$

l.



Parallel form realization with real and complex poles grouped in pairs.

m.

2. Comments

In the previous lecture we introduced the representation of linear digital networks. There are many network structures which can be used to implement a specified transfer function. While all of these structures are mathematically equivalent, they may have different implications for a hardware or software realization of a digital filter.

In this lecture we discuss some of the more common structures, specifically the direct-form, cascade form and parallel form. The notion of canonic structures, i.e. structures utilizing the minimum possible number of delays is also considered. Since delay branches in the flow graph represent memory registers in a hardware implementaion it is generally desireable to minimize the number of delays in a structure. Implementation of a digital filter inevitably involves trade-offs, however, and the importance of minimizing delays may need to be balanced against other considerations.

We also present in this lecture the transposition theorem for signal-flow graphs. This theorem is useful in obtaining new structures. For example we present in this lecture the direct-form II structure and its transpose. Both of these structures are canonic in the number of delays. They differ in the order in which the poles and zeros are implemented, a consideration which is often important when implementing a digital filter with a small register length.

3. Reading

Text: Sections 6.3 (page 300) and 6.4.

4. Problems

Problem 12.1

Consider the discrete-time linear causal system defined by the difference equation

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) + \frac{1}{3}x(n-1)$$

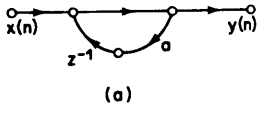
Draw a signal flow graph to implement this system in each of the following forms:

(a) Direct form I.
(b) Direct form II.
(c) Cascade.
(d) Parallel.

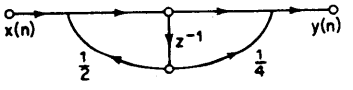
For the cascade and parallel forms use only first-order sections.

Problem 12.2

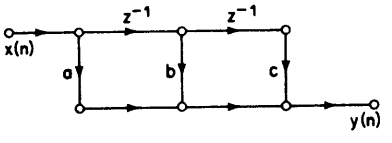
In Figure P12.2-1 (a)-(c) several networks are shown. Determine the transpose of each and verify that in each case the original and transpose networks have the same transfer function.



(a)



(b)

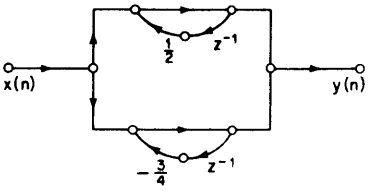


(c)

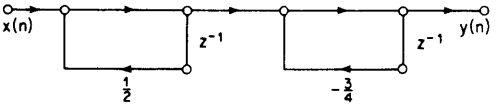
Figure P12.2-1

Problem 12.3

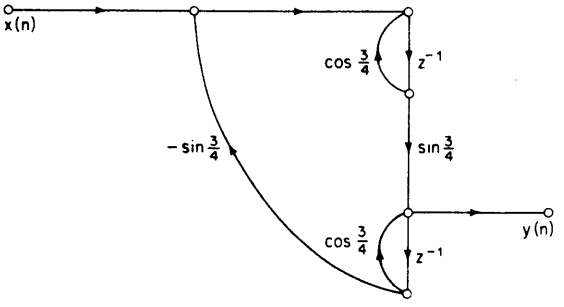
In Figure P12.3-1 (a)-(f) six digital networks are shown. Determine which one of the last five i.e., (b) through (f) has the same transfer function as (a). You should be able to eliminate some of the possibilities by inspection.



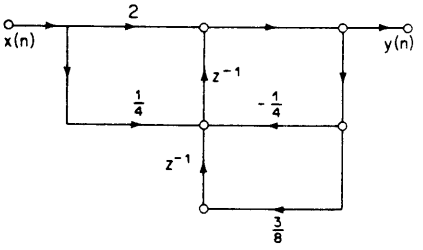
(a)



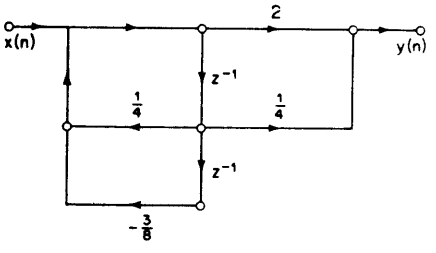
(b)



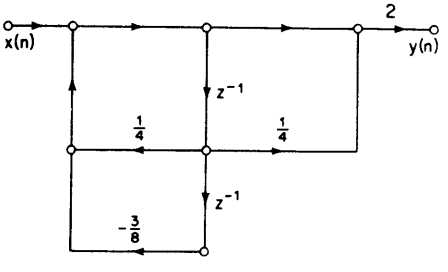
(c)



(d)



(e)



(f)

Figure P12.3-1

Problem 12.4*

The system with transfer function $H(z) = \frac{z^{-1} - a}{1 - az^{-1}}$ is an allpass system, i.e. the frequency response has unity magnitude.

(a) Draw a network realization of this system in direct II form; and indicate in particular the number of delay branches required and the number of branches requiring multiplication by other than +1 or -1.

(b) An alternative implementation is suggested by noting that the difference equation of the allpass system can be expressed as

$$y(n) - ay(n - 1) = x(n - 1) - a\,x(n)$$

or equivalently,

$$y(n)=a\,[y(n - 1) - x(n)] + x(n - 1)$$

Draw a network realization of this equation requiring two delay branches but only one branch with a multiplication by other than +1 or -1.

The primary disadvantage to the network in (b) as compared to that in (a) is that two delay brances are required. In some applications, however, it is necessary to implement a cascade of allpass sections. For N allpass sections it is possible to utilize a realization of each in the form determined in part (b) but using only (N + 1) delay branches. This is accomplished essentially by sharing a delay between sections.

(c) Consider the allpass system with transfer functions

$$H(z) = \frac{z^{-1} - a}{1 - az^{-1}} \qquad \frac{z^{-1} - b}{1 - bz^{-1}}$$

Draw a network realization of this system by "cascading" two networks of the form obtained in part (b) in such a way that only three delay branches are required.

Problem 12.5*

Speech production can be modeled as a linear system representing the vocal cavity, excited by puffs of air released through the vocal cords. In synthesizing speech on a digital computer, one approach is to represent the vocal cavity as a connection of cylindrical acoustic tubes with equal length but with different cross-sectional areas, as depicted in Figure 12.5-1. Let us assume that we want to simulate this system in terms of the volume velocity representing air flow. The input is coupled into the vocal tract through a small constriction, the vocal

cords. We will assume that the input is represented by a change in volume velocity at the left end, but that the boundary condition for traveling waves at the left end is that the net volume velocity must be zero. This is analogous to an electrical transmission line driven by a current source. The output is considered to be the volume velocity at the right end. We assume that each section is lossless.

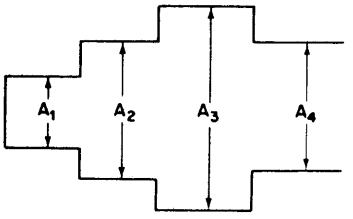


Figure P12.5-1

At each interface between sections a forward-traveling wave is transmitted to the next section with one coefficient and reflected as a backward-traveling wave with a different coefficient. Similarly, a backward-traveling wave arriving at an interface is transmitted with one coefficient and reflected with a different coefficient. Specifically, if we consider a forward-traveling wave f_+ in a tube with cross-sectional area A_1 arriving at the interface with a tube of cross-sectional area A_2, then the forward-traveling wave transmitted is $(1 + \alpha)f_+$ and the reflected wave is αf_+ where

$$\alpha = \frac{A_2 - A_1}{A_1 + A_2}$$

Consider the length of each section to be 3.4 cm with the velocity of sound in air 34,000 cm/s. Draw a digital network that will implement the four-section tube in Figure P12.5, with the output sampled at a 20-kHz rate.

In spite of the lengthy introduction, this is a reasonably straightforward problem. If you find it hard to think in terms of acoustic tubes, think in terms of transmission-line sections with different characteristic impedances. Just as with transmission lines, it is difficult to express the impulse response in closed form. Draw the network directly from physical considerations, in terms of forward- and backward-traveling pulses in each section.

Solution 12.1

(a) Direct Form I (text figure 6.10) corresponds to first implementing the right-hand side of the difference equation (i.e. the zeros) followed by the left-hand side (i.e. the poles). Thus the direct form I for this difference equation is:

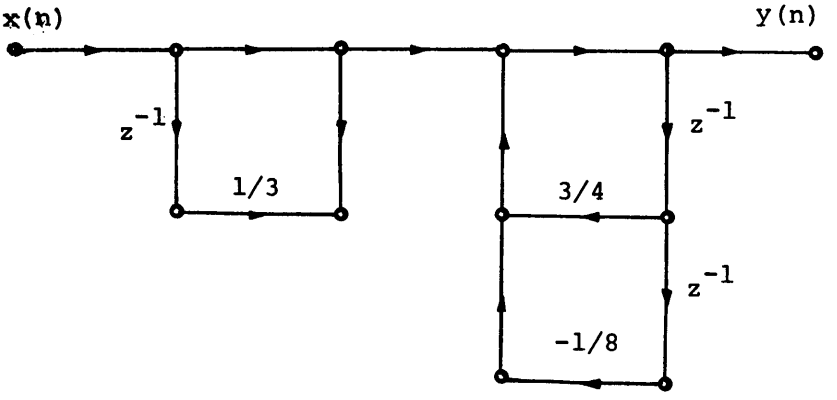


Figure S12.1-1

The flow-graph drawn in this form graphically separates the part implementing the zeros and that implementing the poles. It can alternatively be drawn more efficiently by eliminating some of the branches with unity gain and collapsing some of the nodes. For example an alternative way of depicting the flow-graph of Figure S12.1-1 is:

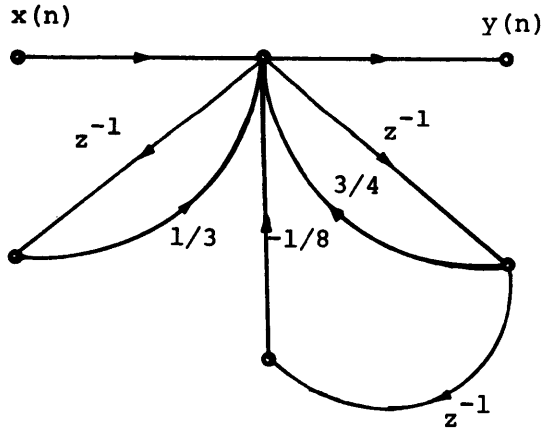


Figure S12.1-2

(b) The direct-form II (text Figure 6.11) corresponds to implementing the poles first, followed by the zeros:

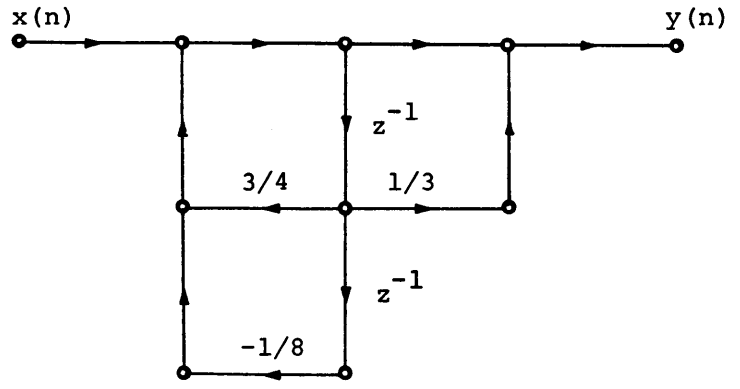


Figure S12.1-3

or alternatively,

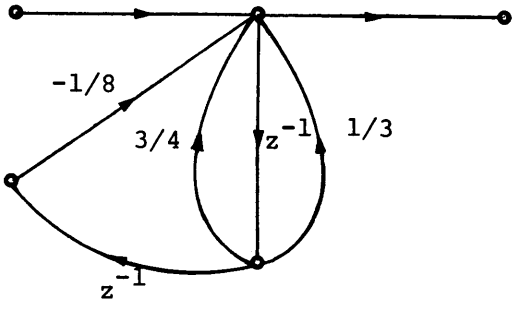


Figure S12.1-4

(c) In the cascade form using first-order sections, we must first factor the system function into a cascade of two first-order systems. Applying the z-transform to both sides of the difference equation,

$$Y(z)\left[1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right] = X(z)\left[1 + \frac{1}{3}z^{-1}\right]$$

or

$$H(z) = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{1 + \frac{1}{3}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})}$$

In developing the cascade form, we can include the zero with either pole and arrange the cascade in either order. For example writing H(z) as

$$H(z) = \left[\frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{4}z^{-1}}\right]\left[\frac{1}{1 - \frac{1}{2}z^{-1}}\right]$$

and using the direct-form II for the first subsection leads to the

cascade form shown in Figure S12.1-5

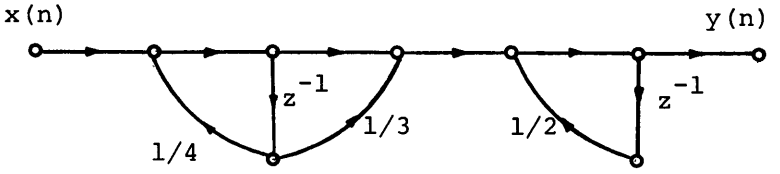


Figure S12.1-5

This flow-graph can also be collapsed somewhat as we have done with those in (a) and (b).

(d) The parallel form corresponds to expanding H(z) in a partial fraction expansion. Thus,

$$H(z) = \frac{10/3}{1 - \frac{1}{2}z^{-1}} + \frac{-7/3}{1 - \frac{1}{4}z^{-1}}$$

leading to the flow graph shown below:

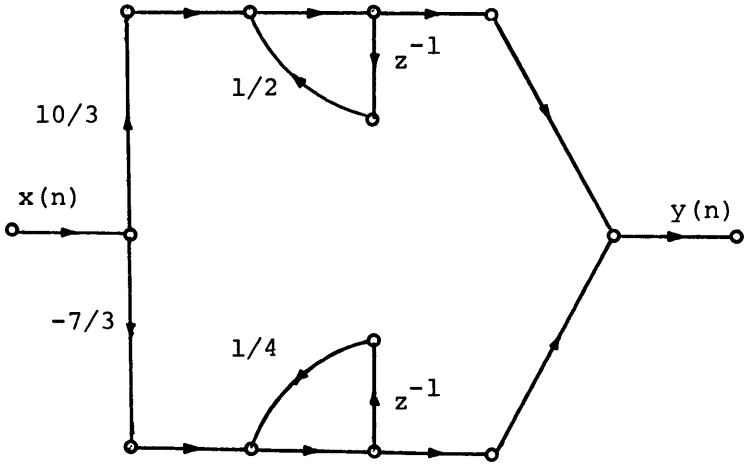


Figure S12.1-6

Solution 12.2

(a) The transpose of this network is:

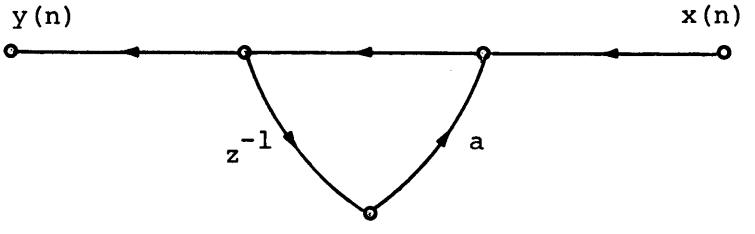


Figure S12.2-1

or, drawing it with the input on the left and the output on the right,

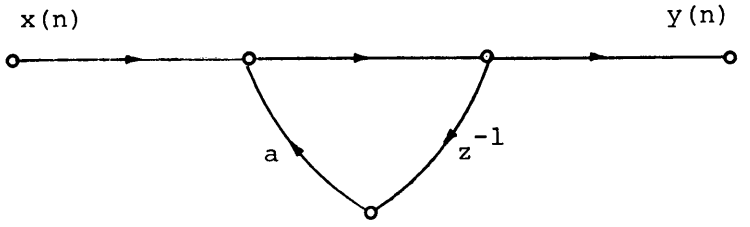


Figure 12.2-2

Since the only effect of the transposition is to interchange the order of the delay and coefficient branches in the feedback path the transfer function is clearly unchanged.

(b) Having worked problem 12.1 we can write down the transfer function of this network by inspection:

$$H(z) = \frac{1 + \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

The transposed network is

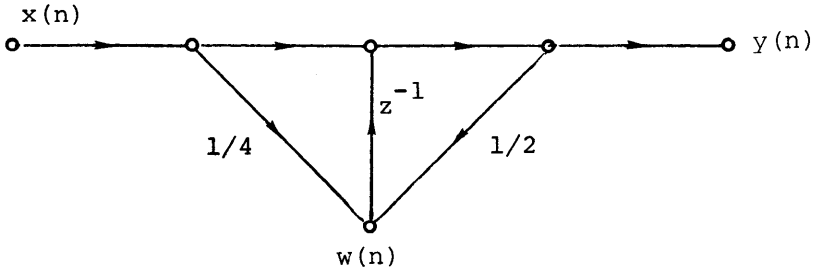


Figure S12.2-3

It is not obvious by inspection (at least not to me) that this transposed network has the same transfer function as the original. However writing the equations for the above flow graph we obtain:

$$w(z) = \frac{1}{2} Y(z) + \frac{1}{4} X(z)$$

and

$$Y(z) = X(z) + z^{-1} W(z)$$

Combining these two equations, we obtain

$$Y(z) = X(z) + z^{-1} \left[\frac{1}{2} Y(z) + \frac{1}{4} X(z)\right]$$

$$\frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

(c) By inspection of the network, we can write that

$$Y(z) = aX(z) + bz^{-1} X(z) + cz^{-2} X(z)$$

or

$$H(z) = a + bz^{-1} + cz^{-2}$$

The transpose network is

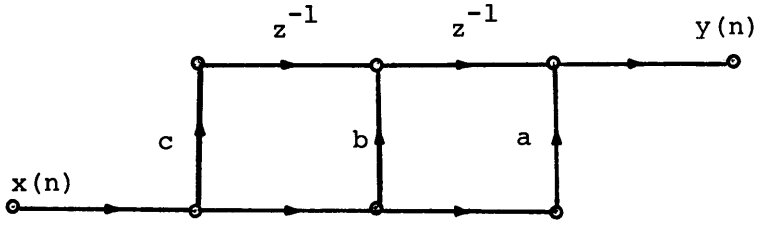


Figure S12.2-4

By inspection of this network we see that

$$Y(z) = cz^{-2} X(z) + bz^{-1} X(z) + aX(z)$$

or $H(z) = a + bz^{-1} + cz^{-2}$

as before.

Solution 12.3

By inspection we observe that for network (a) there is a delay free path with a gain of 2 from input to output, i.e. if $x(n) = \delta(n)$ then $y(0) = 2$. Since this is not true for networks (b) or (c), they are eliminated. Second, by inspection of (a) we note that there are two poles, one at $z = 1/2$ and one at $z = -3/4$, i.e. the denominator of the system function is

$$(1 - \frac{1}{2}z^{-1})(1 + \frac{3}{4}z^{-1}) = (1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}).$$

Since networks (e) and (f) correspond to system functions with denominator polynomial given by

$$(1 - \frac{1}{4}z^{-1} + \frac{3}{8}z^{-2})$$

these networks are eliminated. The remaining possibility, which is the correct answer, is network (d).

Solution 12.4

(a)

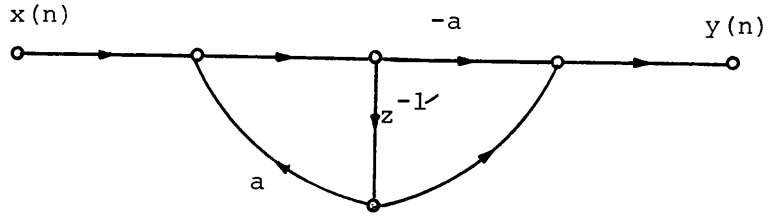


Figure S12.4-1

Note that one delay and two multipliers are required.

(b)

$$y(n) = a\,(y(n-1) - x(n)) + x(n-1)$$

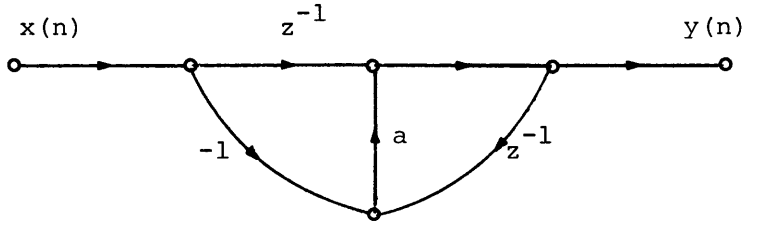


Figure S12.4-2

In this case two delays but only one multiplier are required.

(c) First, let us simply cascade two networks of the form obtained in (b):

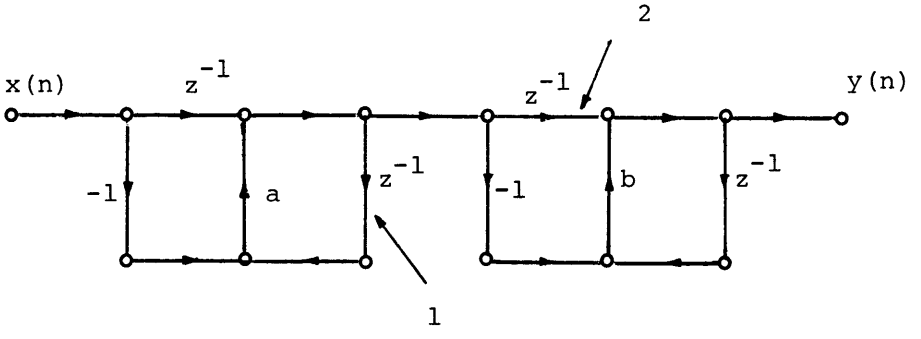


Figure S12.4-3

Now, we note that the input and output of the two branches labelled

"1" and "2" are identical and thus these can be combined into a single delay. For depicting the final network it is also convenient to "flip" the second network above.

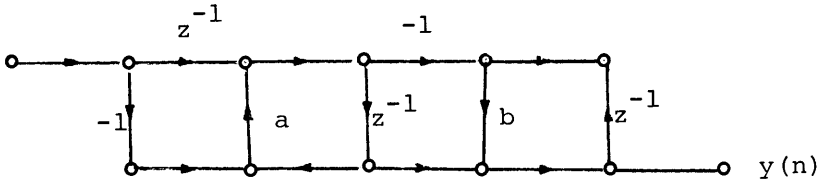


Figure S12.4-4

Solution 12.5

Since each section is 3.4 cm. long and the velocity of sound is 3.4×10^4 cm/sec. it takes 10^{-4} secs. to traverse one section. With a sampling rate of 20 khz the sampling interval is $.5 \times 10^{-4}$ seconds and consequently the length of each section is two sampling intervals, i.e. each can be represented by two delays in cascade. The entire system is linear and so the forward and backward travelling waves add at the boundaries. Let

$$\alpha_{kn} = (A_n - A_k)/(A_n + A_k) \quad .$$

Then the resulting network is

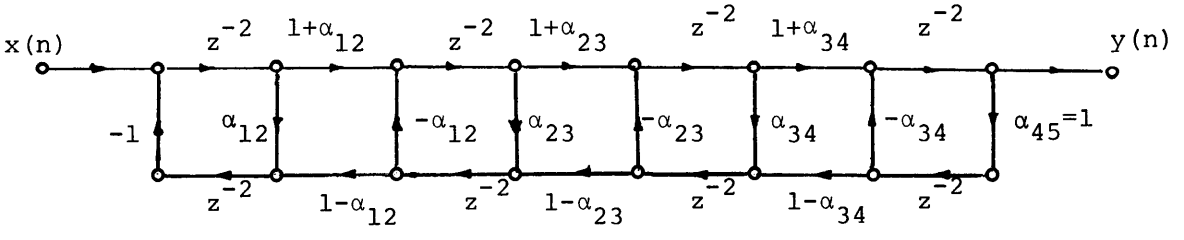
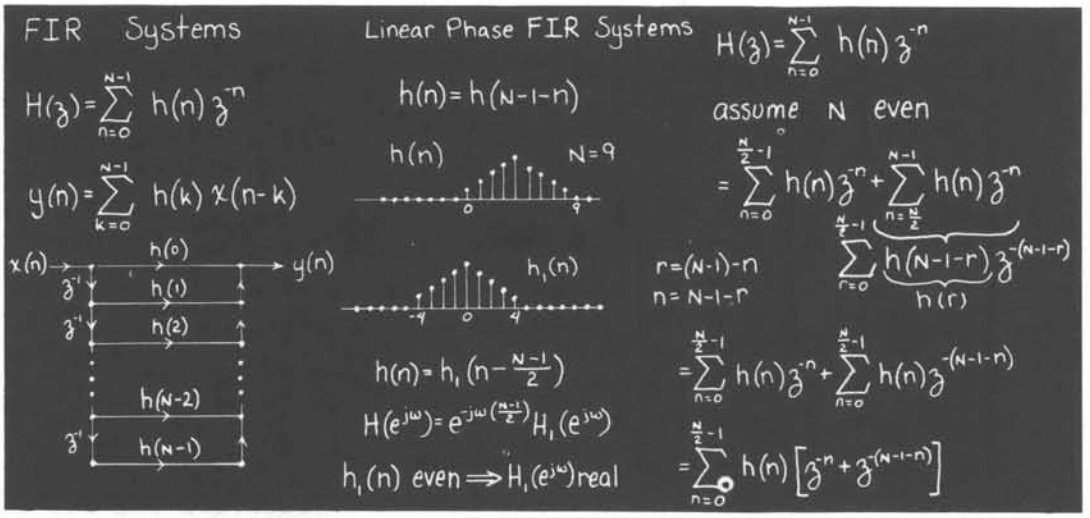


Figure S12.5-1

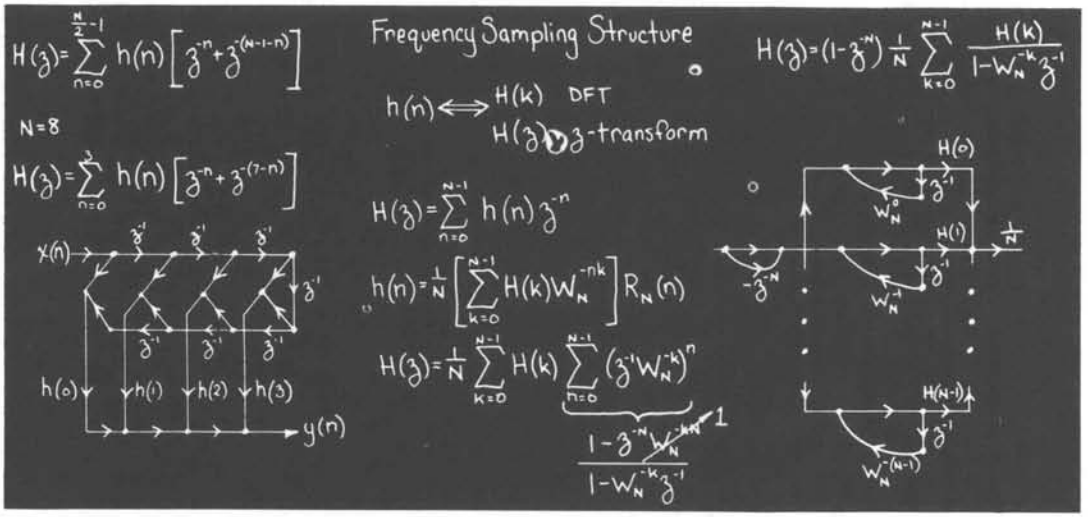
NETWORK STRUCTURES FOR FIR SYSTEMS AND PARAMETER QUANTIZATION EFFECTS IN DIGITAL FILTER STRUCTURES

1. Lesson 13 - 51 minutes

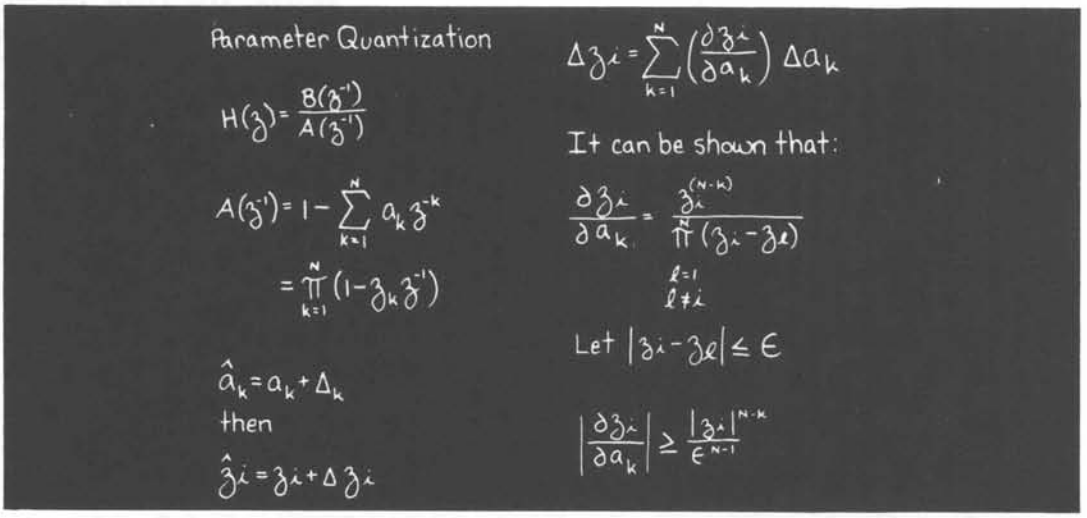
a.

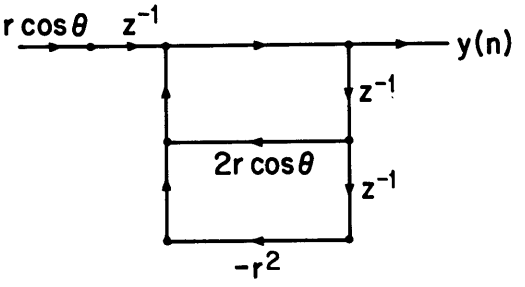


b.



c.

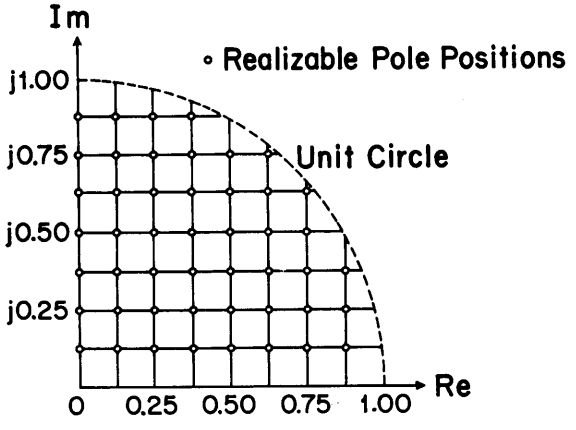




Direct-form implementation of a complex conjugate pole pair.

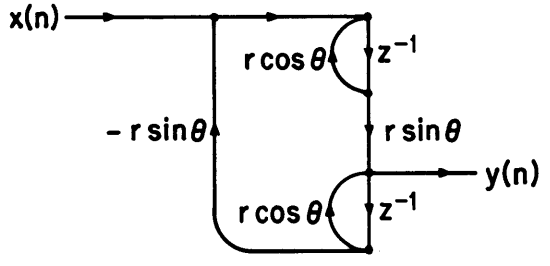
$$H(z) = \frac{r\cos\theta\, z^{-1}}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})}$$

d.



Grid of possible pole locations for the network of viewgraph d when the coefficients are quantized to three bits.

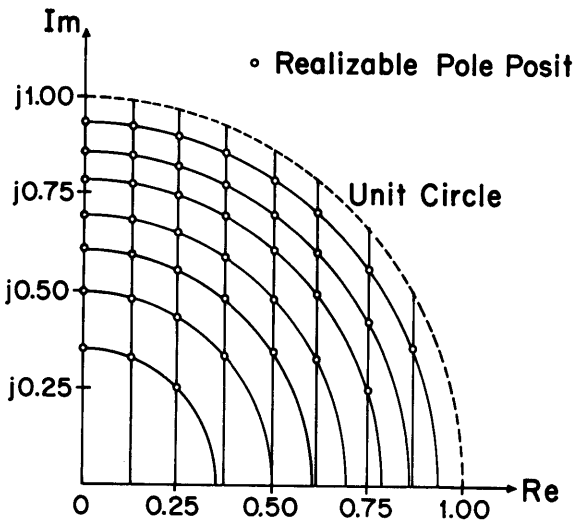
e.



Coupled form implementation of a complex conjugate pole pair. (Note that the transfer function has been corrected. The numerator factor is rsin θ not rcos θ as indicated in the lecture.)

$$H(z) = \frac{r\sin\theta\, z^{-1}}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})}$$

f.



Grid of possible pole locations for the network of viewgraph f when the coefficients are quantized to three bits.

g.

2. Comments

In the previous lecture we discussed several basic network structures for IIR filters. These structures apply also, of course, to the implementation of FIR filters. There are, however, several additional structures which apply specifically to FIR systems, two of which are discussed in this lecture.

The first of these structures applies specifically to linear phase FIR systems. FIR systems can be designed to have exactly linear phase by constraining the unit sample response to be symmetrical. With this symmetry there are only $\frac{N}{2}$ (for N even) or $\frac{N+1}{2}$ (for N odd) independent coefficients and such filters can be implemented with a structure requiring only $\frac{N}{2}$ or $\frac{N+1}{2}$ multiplications. The second FIR structure discussed in this lecture is referred to as the frequency-sampling structure because the coefficients of this structure are samples of the frequency response of the system.

The importance of different structures for digital filters is tied very closely to the considerations involved in a hardware implementation. One such consideration is the effect of finite register length. As an introduction to this issue we discuss briefly in this lecture the relation between structures with regard to parameter quantization effects. The primary result stressed is that the sensitivity of pole and zero locations to parameter quantization tends to be higher for a direct form structure than for a cascade form

structure. Even within the cascade structure there is flexibility with regard to how the pole-zero pairs are implemented, resulting in different parameter quantization effects. In particular, coefficient quantization constrains the poles (and zeros) to lie on a grid in the z-plane. The location and density of the grid points is determined by the amount of parameter quantization and the form of the network structure.

3. Reading

Text: Sections 6.5 (page 313) and 6.8.

4. Problems

Problem 13.1

H(z) represents the system function for an FIR linear system, and is given by

$$H(z) = (1+ \tfrac{1}{2}z^{-1})\ (1 + 2z^{-1})\ (1 - \tfrac{1}{4}z^{-1})\ (1 - 4z^{-1})$$

Draw a flow-graph implementation of the system in each of the following forms:

(i) cascade form
(ii) direct form
(iii) linear-phase form
(iv) frequency-sampling form

Problem 13.2

We wish to implement a digital oscillator, i.e., a digital filter for which the unit-sample response is of the form

$$h(n) = A \cos(\omega_0 n + \phi)\ u(n) \qquad (13.2\text{-}1)$$

Shown in Figure P13.2-1 are a direct form and a coupled form second order filter.

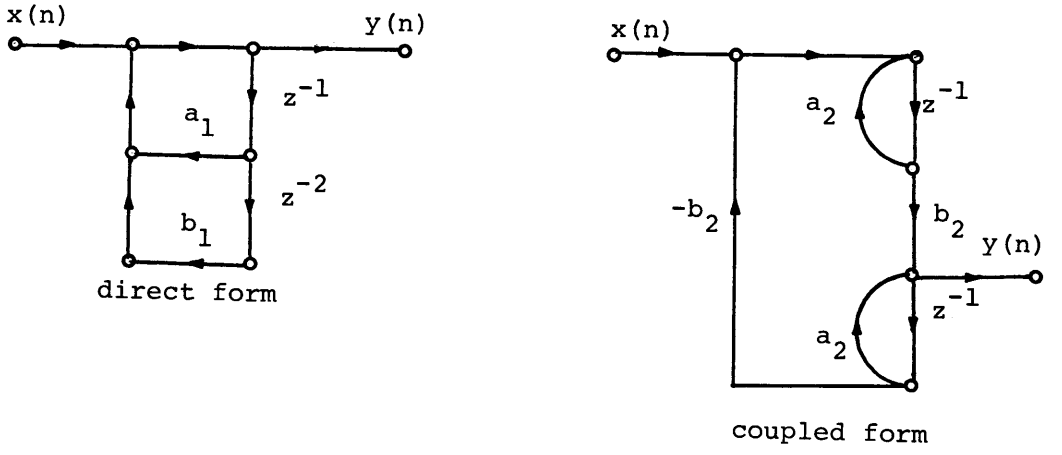


Figure P13.2-1

(a) Assuming that there is no parameter quantization, determine the coefficients a_1 and b_1 so that the unit-sample response of the direct form filter will be of the form of Eq. (13.2-1) with $\omega_0 = \frac{\pi}{4}$.

(b) Again, assuming that there is no parameter quantization determine the coefficients a_2 and b_2 so that the unit-sample response of the coupled form filter will be of the form of Eq. (13.2-1) with $\omega_0 = \frac{\pi}{4}$. Although the amplitude A and phase shift ϕ will be different for the direct form and coupled form, this difference is not important for this problem.

(c) Now, let us quantize the coefficients for the two filters. In particular, the fractional part of the absolute value of each coefficient will be represented by, at most three bits. The value of the quantized coefficient can be obtained as follows:

(i) multiply the coefficient by 8
(ii) discard the fractional part of the product
(iii) divide the result in (ii) by 8.

Determine whether, after quantization the unit-sample response of each filter will still be of the form of Eq. (13.2-1). If yes, determine the new value of ω_0.

Problem 13.3

A causal linear-phase FIR system has the property that $h(n) = h(N - 1 - n)$ for $n = 0,1\ldots, N - 1$. This symmetry constraint was used in Sec. 6.5.3 of the text to show that systems satisfying this constraint have linear phase corresponding to a delay of $(N - 1)/2$ samples. This constraint results in a significant simplication of the frequency-sampling realization of Eqs. (4.48) and (4.50) of the text.

(a) Using the above linear-phase constraint, show that, for N even, $\tilde{H}(N/2) = 0$.

(b) Consider H(k) expressed in the form $H(k) = \tilde{H}_a(k)e^{j\theta(k)}$ where $\tilde{H}_a(k)$ is real. Determine an expression for $\theta(k)$ for $k = 0, 1,\ldots, N - 1$ that is valid for N even. You may find it helpful to refer to the results in Sec. 6.5.3 of the text.

(c) Assume that $\tilde{H}_a(k)$ defined in (b) is position i.e. that it corresponds to the magnitude of $\tilde{H}(k)$. Using the results of part (a) and (b), show that for h(n) linear phase and with N even, Eqs. (4.49) and (4.50) of text can be simplified to

$$H(z) = \frac{1 - z^{-N}}{N}\left[\sum_{k=1}^{(N/2)-1} \frac{(-1)^k|\tilde{H}(k)|2\cos(\pi k/N)(1-z^{-1})}{1 - 2z^{-1}\cos(2\pi k/N) + z^{-2}} + \frac{\tilde{H}(0)}{1 - z^{-1}}\right].$$

(We have assumed for convenience that r = 1.)

(d) Draw a flow-graph representation of the system function derived in part (c).

NETWORK STRUCTURES FOR FIR SYSTEMS AND PARAMETER QUANTIZATION EFFECTS IN DIGITAL FILTER STRUCTURES

Solution 13.1

(i) Since H(z) has only real zeros we will use only first-order sections in the cascade form. Then Figure S13.1-1 represents one possible ordering for these sections.

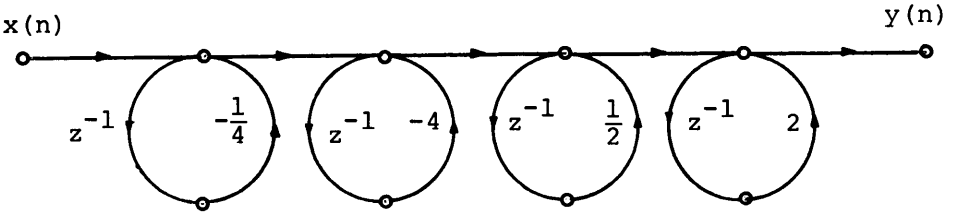


Figure S13.1-1

(ii) For the direct form we first express H(z) as

$$H(z) = 1 - \frac{7}{4}z^{-1} - \frac{69}{8}z^{-2} - \frac{7}{4}z^{-3} + z^{-4}$$

The direct-form structure is then as shown in Figure S13.1-2

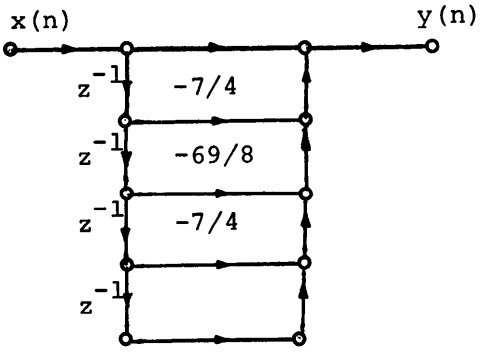


Figure 13.1-2

(iii) Since the unit-sample response is symmetrical the filter is in fact linear phase. The linear phase form is as shown in Figure S13.1-3.

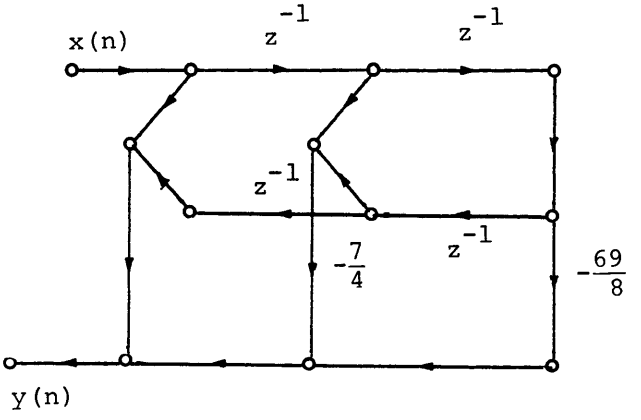


Figure S13.1-3

(iv) The coefficients in the frequency-sampling structure are the samples of the frequency response equally spaced in frequency. To evaluate these frequency samples it is convenient to rewrite H(z) in the form

$$H(z) = z^{-2}\left[z^2 - \frac{7}{4}z - \frac{69}{8} - \frac{7}{4}z^{-1} + z^{-2}\right]$$

or

$$H(z) = z^{-2}\left[(z^2+z^{-2}) - \frac{7}{4}(z+z^{-1}) - \frac{69}{8}\right].$$

Then

$$\tilde{H}(0) = H(z)\Big|_{z=1} = -\frac{81}{8}$$

$$\tilde{H}(1) = H(z)\Big|_{z=e^{j\frac{2\pi}{5}}} = e^{-j\frac{4\pi}{5}}\left(2\cos\frac{4\pi}{5} - \frac{7}{2}\cos\frac{2\pi}{5} - \frac{69}{8}\right)$$

$$\tilde{H}(2) = H(z)\Big|_{z=e^{j\frac{4\pi}{5}}} = e^{-j\frac{8\pi}{5}}\left[2\cos\frac{8\pi}{5} - \frac{7}{2}\cos\frac{4\pi}{5} - \frac{69}{8}\right]$$

Because of the conjugate symmetry of the Fourier transform,

$$\tilde{H}(3) = \tilde{H}^*(5-3) = \tilde{H}^*(2) = e^{j\frac{8\pi}{5}}\left[2\cos\frac{8\pi}{5} - \frac{7}{2}\cos\frac{4\pi}{5} - \frac{69}{8}\right]$$

$$\tilde{H}(4) = \tilde{H}^*(5-4) = \tilde{H}^*(1) = e^{j\frac{4\pi}{5}}\left[2\cos\frac{4\pi}{5} - \frac{7}{2}\cos\frac{2\pi}{5} - \frac{69}{8}\right].$$

The frequency sampling structure, in the form of chalkboard (b) lecture 13, is then as shown below:

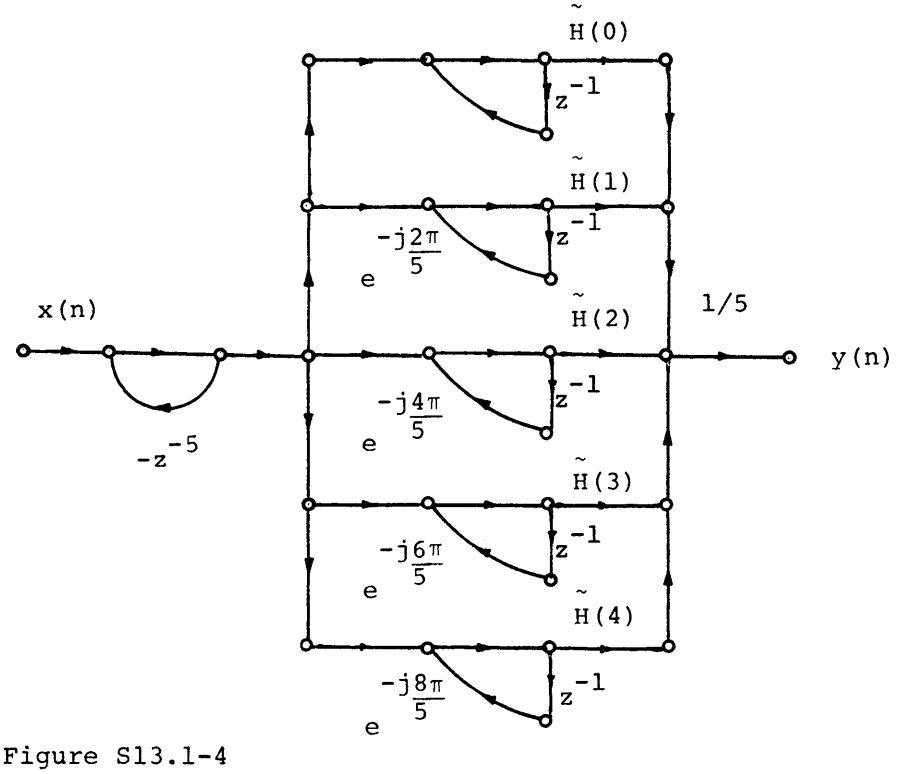


Figure S13.1-4

In this form the sections involve complex coefficients. By utilizing the conjugate symmetry of the frequency samples the network can be rearranged in terms of second-order sections with real coefficients.

Specifically the transfer function corresponding to the recursive part of the network of Figure S13.1-4 is given by

$$G(z) = \frac{\tilde{H}(0)}{1-z^{-1}} + \left[\frac{\tilde{H}(1)}{1-e^{-j\frac{2\pi}{5}}z^{-1}} + \frac{\tilde{H}(4)}{1-e^{-j\frac{8\pi}{5}}z^{-1}}\right] + \left[\frac{\tilde{H}(2)}{1-e^{-j\frac{4\pi}{5}}} + \frac{\tilde{H}(3)}{1-e^{-j\frac{6\pi}{5}}}\right]$$

The terms paired in brackets are complex conjugates, i.e.,

$$\tilde{H}(1) = \tilde{H}^{*}(4) \text{ and } e^{-j\frac{8\pi}{5}} = e^{j2\pi}e^{-j\frac{8\pi}{5}} = e^{j\frac{2\pi}{5}}$$

$$\tilde{H}(2) = \tilde{H}^{*}(3) \text{ and } e^{-j\frac{6\pi}{5}} = e^{j\frac{4\pi}{5}} .$$

Continuing these complex conjugate terms, G(z) can be expressed as

$$G(z) = \frac{\tilde{H}(0)}{1-z^{-1}} + \frac{2\mathrm{Re}[\tilde{H}(1)]-2z^{-1}\mathrm{Re}[\tilde{H}(1)e^{j\frac{2\pi}{5}}]}{1-2z^{-1}\cos\frac{2\pi}{5}+z^{-2}}$$

$$+ \frac{2\mathrm{Re}[\tilde{H}(2)]-2z^{-1}\mathrm{Re}[\tilde{H}(2)e^{j\frac{4\pi}{5}}]}{1-2z^{-1}\cos\frac{4\pi}{5}+z^{-2}}$$

leading to the structure shown in Figure S13.1-5.

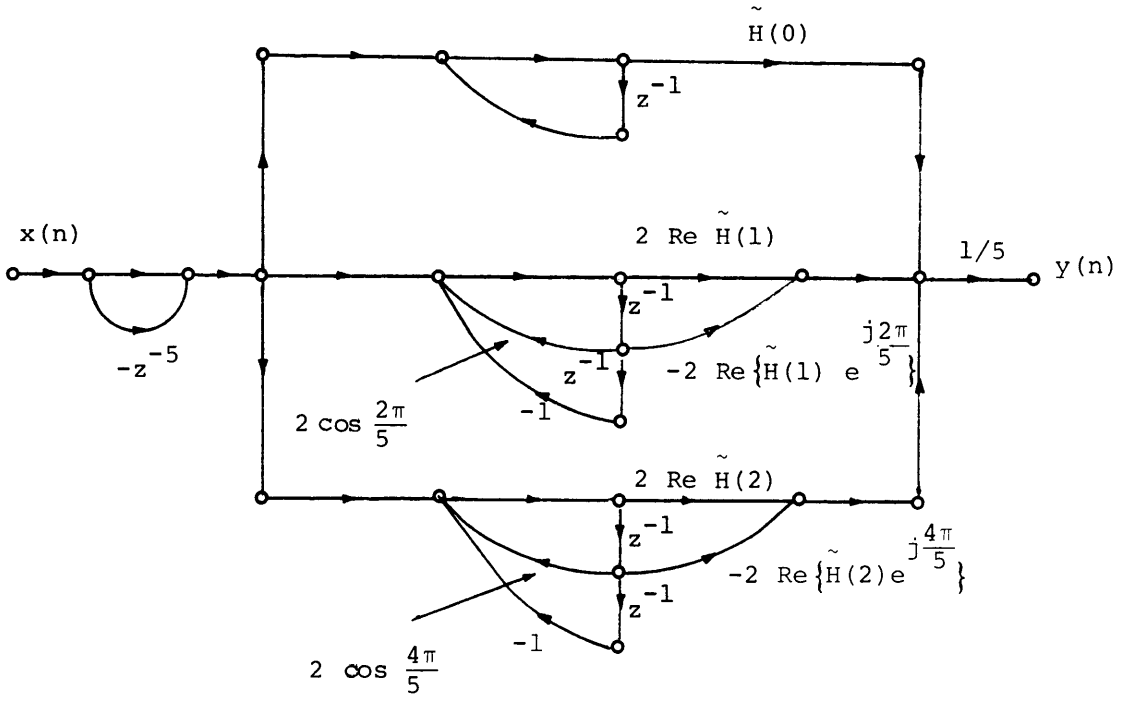


Figure S13.1-5

Solution 13.2

(a) The form of the desired transfer function is easily obtained by expressing h(n) in the form

$$h(n) = \frac{A}{2}\left[e^{j\phi}e^{j\omega_0 n}u(n) + e^{-j\phi}e^{-j\omega_0 n}u(n)\right]$$

Thus

$$H(z) = \frac{A}{2}\left[\frac{e^{j\phi}}{1-e^{j\omega_0}z^{-1}} + \frac{e^{-j\phi}}{1-e^{-j\omega_0}z^{-1}}\right] = A\left[\frac{\cos\phi - z^{-1}\cos(\omega_0-\phi)}{1-2z^{-1}\cos\omega_0+z^{-2}}\right]$$

Since the direct form filter does not contain any zeros, A and ϕ must be such that $A\cos\phi = 1$ and $\cos(\omega_0-\phi)=0$. Then for $\omega_0=\frac{\pi}{4}$ H(z) is

$$H(z) = \frac{1}{1-\sqrt{2}\,z^{-1}+z^{-2}}$$

Thus,

$$a_1 = \sqrt{2}$$

$$b_1 = -1$$

(b) The form of the transfer function for the coupled form network is (see Problem 11.3)

$$H_2(z) = b_2 z^{-1}/[1-2a_2 z^{-1} + (a_2^2+b_2^2)z^{-2}]$$

Thus to obtain a unit-sample response of the desired form, $A\cos\phi = 0$ and $A\cos(\omega_0-\phi)=-1$. The coefficients a_2 and b_2 are given by

$$a_2 = \cos\omega_0 = \frac{\sqrt{2}}{2}$$

$$b_2 = \sin\omega_0 = \frac{\sqrt{2}}{2}$$

(c) For the direct form filter since the coefficient b_1 is unity, it is not affected by quantization. For the coefficient a_1, the quantized value is

$$\hat{a}_1 = \frac{11}{8} = 1.375$$

The resulting transfer function is

$$H_1(z) = \frac{1}{1 - \frac{11}{8} z^{-1}+z^{-2}} \quad .$$

Since this is still in the form of the desired H(z), the unit-sample response will still be of the form of Eq. (13.2-1). With the quantized coefficient, $2\cos\omega_0$ is now equal to 11/8 so that $\omega_0 \tilde{=} .26\pi$

For the coupled form, the quantized coefficients are given by

$$\hat{a}_2 = \hat{b}_2 = \frac{5}{8} = 0.625$$

Thus $H_2(z)$ with quantized coefficients becomes

$$H_2(z) = (5/8)z^{-1}/[1 - \frac{5}{4}z^{-1} + \frac{25}{32}z^{-2}] .$$

In comparing this with the desired H(z) we note that since the coefficient of z^{-2} is not unity, the resulting unit-sample response will not be of the desired form. In particular, the unit-sample response will be of the form of a damped sinusoidal sequence, corresponding to the fact that the poles of the coupled form system with quantized coefficients have moved inside the unit circle. In contrast, the poles of the direct form system with quantized coefficients remain on the unit circle but are displaced in angle. These results are of course consistent with the differences in the quantization grids for the two structures, as illustrated in Figures 6.49 and 6.51 of the text.

Solution 13.3

$$(a)\ \tilde{H}(k) = \sum_{n=0}^{N-1} h(n)\ W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n)\ W_N^{nk} + \sum_{n=0}^{\frac{N}{2}-1} h(N-1-n)\ W_N^{(N-1-n)k}$$

which, because of the symmetry constraint becomes

$$\tilde{H}(k) = \sum_{n=0}^{\frac{N}{2}-1} h(n)\ [W_N^{nk} + W_N^{(N-1-n)k}]$$

For $k = \frac{N}{2}$,

$$\tilde{H}\left(\frac{N}{2}\right) = \sum_{n=0}^{\frac{N}{2}-1} h(n)\ [e^{-j\pi n} + e^{-j\pi(N-1-n)}]$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n)\ \underbrace{[e^{-j\pi n} - e^{j\pi n}]}_{0}$$

$$= 0\ .$$

(b) $$\tilde{H}(k) = \sum_{n=0}^{\frac{N}{2}-1} h(n)\ [W_N^{nk} + W_N^{(N-1-n)k}]$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n)\ W_N^{(\frac{N-1}{2})k}\ [W_N^{k(n-\frac{N-1}{2})} + W_N^{-k(n-\frac{N-1}{2})}]$$

$$= W_N^{(\frac{N-1}{2})k} \sum_{n=0}^{\frac{N}{2}-1} 2h(n)\ \cos\left[\frac{2\pi k}{N}\left(n-\frac{N-1}{2}\right)\right]\ .$$

The summation in the above expression is real. Let us assume for convenience that it is also positive. If it is not, then for those values of k for which it is negative, an additional phase of π will be added. Then, with this assumption,

$$e^{j\theta(k)} = W_N^{(\frac{N-1}{2})k}$$

or

$$\theta(k) = -\frac{\pi}{N}(N-1)k = -\pi k + \frac{\pi k}{N}$$

(c) $$H_k(z) = \frac{\cos(\frac{\pi k}{N} - \pi k) - z^{-1}\cos(\frac{-\pi k}{N} - \pi k)}{1-2z^{-1}\cos(\frac{2\pi k}{N}) + z^{-2}}$$

$$= \frac{(-1)^k \cos\frac{\pi k}{N}(1-z^{-1})}{1-2z^{-1}\cos(\frac{2\pi k}{N}) + z^{-2}} \quad .$$

Thus, from Equation (4.49) of the text.

$$H(z) = \frac{1-z^{-N}}{N}\left[\sum_{k=1}^{\frac{N}{2}-1} \frac{(-1)^k|\tilde{H}(k)|2\cos(\pi k/N)(1-z^{-1})}{1-2z^{-1}\cos(2\pi k/N) + z^{-2}} + \frac{\tilde{H}(0)}{1-z^{-1}}\right] .$$

(d)

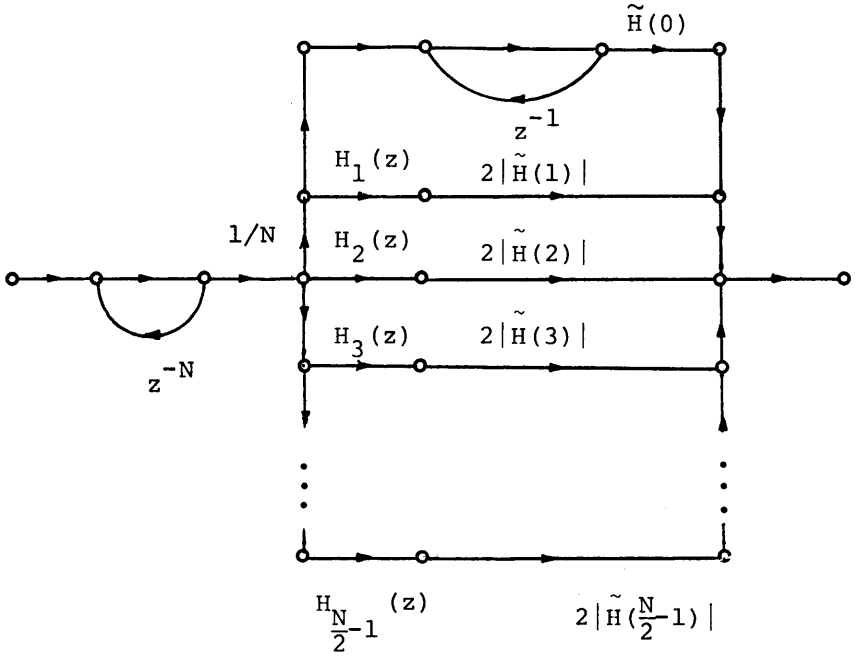


Figure S13.3-1

where the subnetworks $H_k(z)$ are of the form

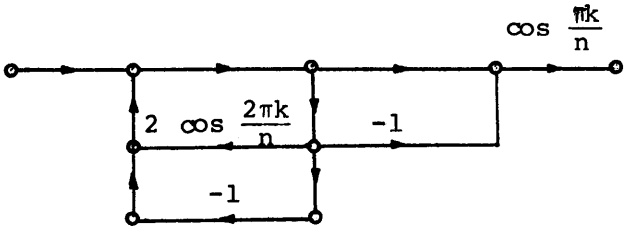
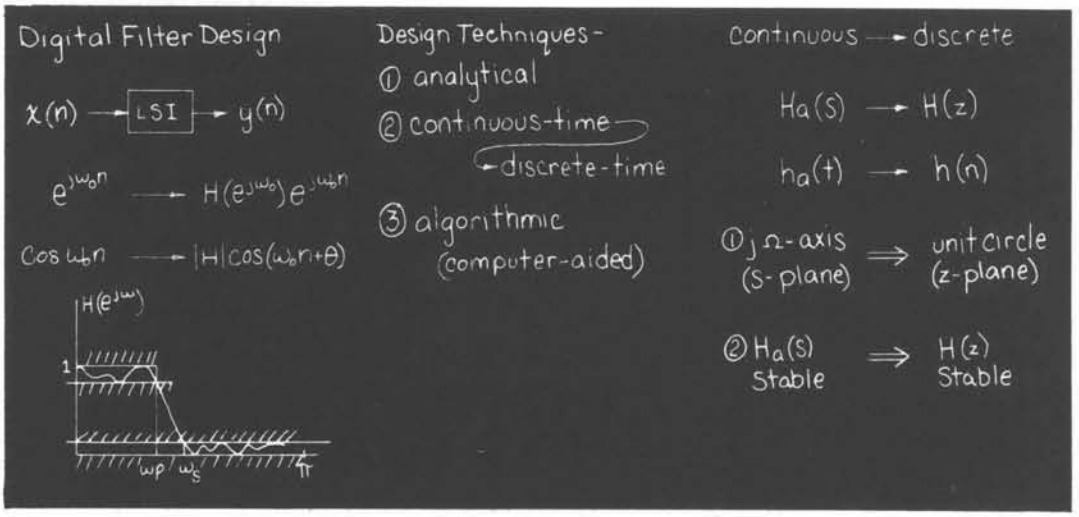


Figure S13.3-2

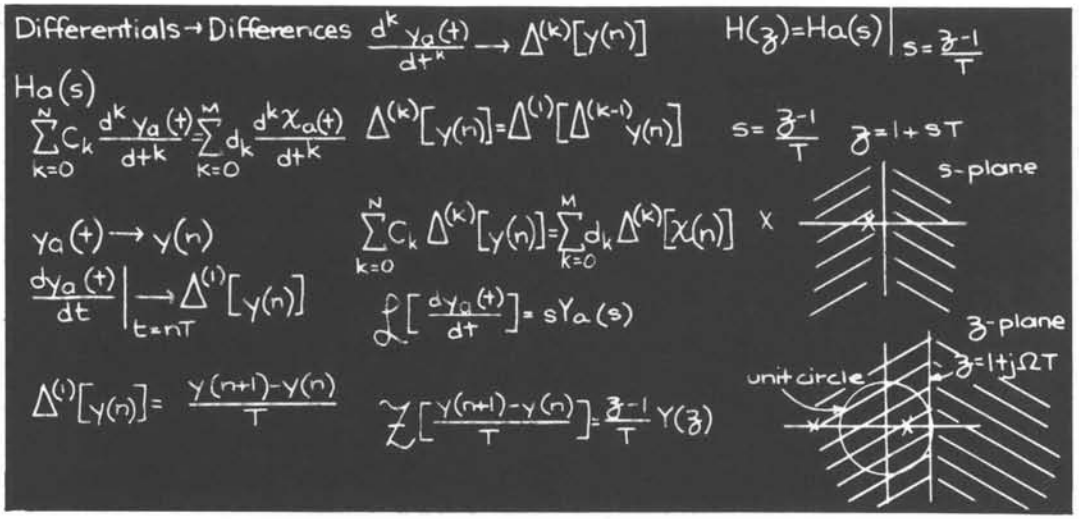
DESIGN OF IIR DIGITAL FILTERS - PART 1

1. Lecture 14 - 47 minutes

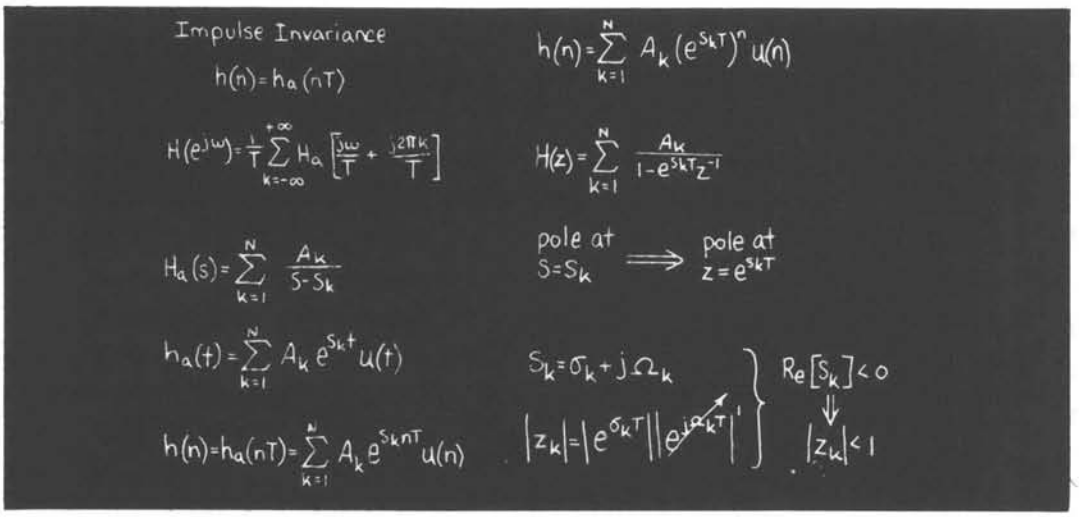
a.

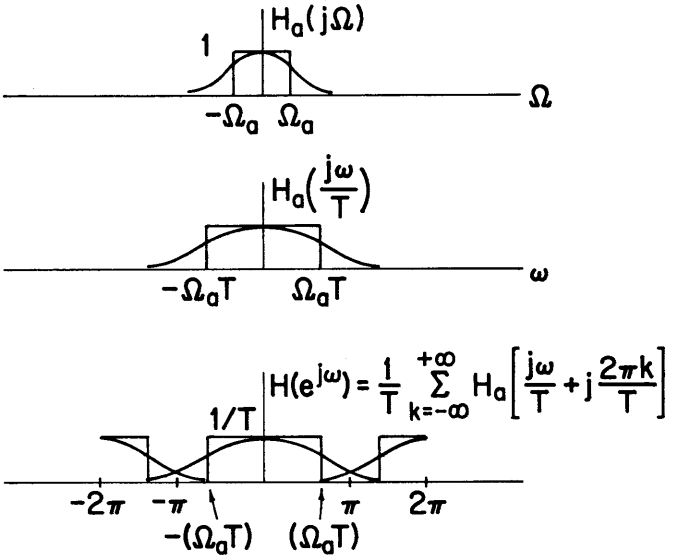


b.



c.





d. An analog frequency response and the corresponding digital frequency response obtained through impulse invariance.

2. Comments

With this lecture we begin the discussion of digital filter design techniques. The concept of frequency selective filtering for discrete-time signals is identical to that for continuous-time signals and stems from the fact that complex exponentials or sinusoids are eigenfunctions of linear shift-invariant systems. Just as with analog filters, ideal frequency response characteristics cannot be achieved exactly and must be approximated.

Design methods for analog filters have a long history and a variety of elegant design procedures have been developed. Many of the most useful digital filter design techniques are directed at transforming these analog filter designs to digital filter designs, thus taking advantage of a rich collection of available filter designs.

In this lecture two such transformation procedures are discussed. The first corresponds to approximating the linear constant coefficient differential equation for the analog filter by a linear constant coefficient difference equation by replacing derivative by differences. As we see, this transformation is not a useful one since it does not map the analog frequency response onto the unit circle and does not guarantee that a stable analog filter will yield a stable digital filter.

The second transformation discussed is the use of impulse invariance, corresponding to obtaining the discrete-time unit sample response by sampling the analog impulse response. Except for the effect of aliasing the digital frequency response obtained is a scaled replica of the analog frequency response.

3. Reading

Text: Sections 7.0 (page 403) and 7.1 up to example 7.3.
(Example 7.3 will be covered in lecture 16.)

4. Problems

Problem 14.1

Consider an analog filter for which the input $x_a(t)$ and output $y_a(t)$ are related by the linear constant-coefficient differential equation

$$\frac{dy_a(t)}{dt} + 0.9\, y_a(t) = x_a(t)$$

A digital filter is obtained by replacing the first derivative by the first forward difference so that with x(n) and y(n) denoting the input and output of the digital filter,

$$\left[\frac{y(n+1) - y(n)}{T}\right] + 0.9\, y(n) = x(n)$$

Throughout this problem the digital filter is assumed to be causal.

(a) Determine and sketch the magnitude of the frequency response of the analog filter.

(b) Determine and sketch the magnitude of the frequency response of the digital filter for T = 10/9.

(c) Determine the range of values of T for which the digital filter is unstable. (Note that the analog filter is stable.)

Problem 14.2

$h_a(t)$ denotes the impulse response of an analog filter and is given by

$$h_a(t) = \begin{cases} e^{-0.9t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Let h(n) denote the unit sample response and H(z) denote the system function for the digital filter designed from this analog filter by impulse invariance, i.e. with

$$h(n) = h_a(nT)$$

Determine H(z), including T as a parameter, and show that for any positive value of T, the digital filter is stable. Indicate also whether the digital filter approximates a lowpass filter or a highpass filter.

Problem 14.3

We now wish to design a digital filter from the analog filter of problem 14.2 using step invariance. Let $s_a(t)$ denote the step response of the analog filter of problem 4.2 and $s(n)$ the step response of the digital filter, so that

$s(n) = s_a(nT)$

(a) Determine $s_a(t)$

(b) Determine $s(n)$

(c) Determine $H(z)$, the system function of the digital filter. Note, in particular, that it is not the same as the system function obtained in problem 14.2 by using impulse invariance.

Problem 14.4

The system function $H_a(s)$ of an analog filter is

$$H_a(s) = \frac{s}{(s+1)(s+2)}$$

Determine the system function $H(z)$ of the digital filter obtained from this analog filter by impulse invariance.

Problem 14.5*

An ideal bandlimiting differentiator with delay τ is defined by the frequency response

$$H_a(j\Omega) = \begin{cases} j\Omega e^{-j\Omega\tau} & |\Omega| \leq \Omega_c \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the frequency response $H_d(e^{j\omega})$ of the digital filter obtained from this analog filter by impulse invariance. Assume that $\frac{\pi}{T} > \Omega_c$.

(b) Let $\hat{h}_d(n)$ denote the unit sample response of the filter determined (a) with $\tau = 0$. For certain values of τ, $h_d(n)$ can be expressed as $\hat{h}_d(n)$ delayed, i.e.

$h_d(n) = \hat{h}_d(n - n_\tau)$

where n_τ is an integer. Determine this set of values of τ and the resulting delay n_τ.

DESIGN OF IIR DIGITAL FILTERS - PART 1

Solution 14.1

(a) Applying the Laplace transform to both sides of the differential equation we obtain

$$Y_a(s)\,[\,s + 0.9\,] = X_a(s)$$

or

$$\frac{Y_a(s)}{X_a(s)} = \frac{1}{s + 0.9} = H(s)$$

Thus

$$H_a(j\Omega) = \frac{1}{j\Omega + 0.9}$$

and

$$|H_a(j\Omega)|^2 = \frac{1}{\Omega^2 + (0.9)^2}$$

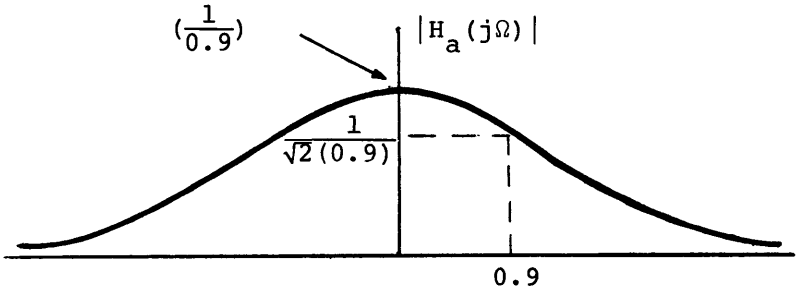


Figure S14.1-1

(b) Applying the z-transform to both sides of the difference equation we obtain

$$Y(z)\left[\frac{z - 1}{T}\right] + 0.9\,Y(z) = X(z)$$

or

$$H(z) = \frac{Y(z)}{X(z)} = {}^{T}\Big/\left[\,z + (0.9T - 1)\right]$$

with $T = \frac{10}{9}$,

$$H(z) = \frac{10/9}{z} = \frac{10}{9}\,z^{-1}$$

and

$$|H(e^{j\omega})| = \frac{10}{9}$$

For this particular choice for T the frequency response is constant, independent of frequency in contrast to the analog filter, which is a lowpass filter. While this is a particularly severe example of the effect of transforming an analog filter to a digital filter by replacing derivatives by differences, it emphasizes the fact that the frequency response of the resulting digital filter will in general be severely distorted from that of the original analog filter.

(c) From the system function determined in part (b), the pole is at $z = 1 - 0.9T$. Assuming T to be positive, the pole is outside of the unit circle for $T \geq 20/9$.

Solution 14.2

$$h(n) = h_a(nT) = e^{-0.9nT}\, u(n)$$

$$= \left(e^{-0.9T}\right)^n u(n)$$

Thus

$$H(z) = \frac{1}{1 - e^{-0.9T} z^{-1}}$$

The frequency response is given by

$$H(e^{j\omega}) = \frac{1}{1 - e^{-0.9T} e^{-j\omega}}$$

The magnitude of which is sketched in figure S14.2-1.

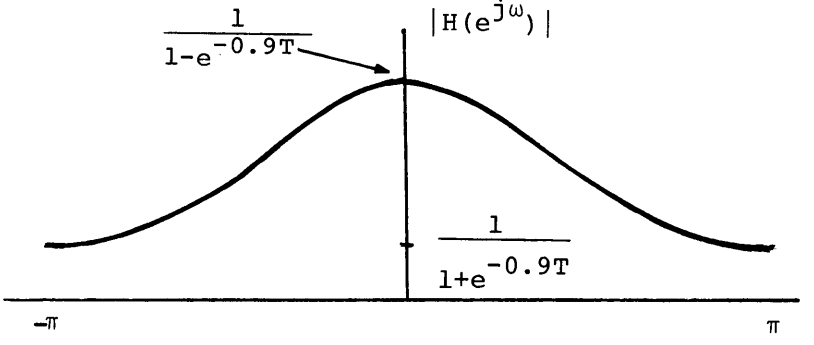


Figure S14.2-1

Thus the digital filter approximates a lowpass filter. Since the pole is located at $z = e^{-0.9T}$ the pole is inside the unit circle and hence the system is stable for $T > 0$.

Solution 14.3

(a) The step response of the analog filter is the integral of its impulse response, i.e.

$$s_a(t) = \int_{-\infty}^{t} h_a(\tau)\, d\tau$$

Hence for $t < 0$ $s_a(t) = 0$ and for $t \geq 0$

$$s_a(t) = \int_{0}^{t} e^{-0.9\tau}\, d\tau = \frac{1}{0.9}\left[1 - e^{-0.9t}\right]$$

(b) $s(n) = s_a(nT) = \frac{1}{0.9}\left[1 - e^{-0.9nT}\right] u(n)$

(c) Let $S(z)$ denote the z-transform of $s(n)$. Then, since the z-transform of a unit step is $\frac{1}{1 - z^{-1}}$, $H(z)$ is given by

$$H(z) = (1 - z^{-1})\, S(z)$$

From (b),

$$S(z) = \frac{1}{0.9}\left[\frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-0.9T} z^{-1}}\right]$$

and

$$H(z) = \frac{1}{0.9}\left[1 - \frac{(1 - z^{-1})}{1 - e^{-0.9T} z^{-1}}\right]$$

$$= \frac{1}{0.9}\left[\frac{z^{-1}(1 - e^{-0.9T})}{1 - e^{-0.9T} z^{-1}}\right]$$

Solution 14.4

The most straightforward procedure is to expand $H_a(s)$ in a partial-fraction expansion and utilize the relationship between $H_a(s)$ and $H(z)$ indicated on chalkboard (c) of lecture 14. Thus,

$$H_a(s) = \frac{A_1}{s + 1} + \frac{A_2}{s + 2}$$

where

$$A_1 = H_a(s)(s + 1)\Big|_{(s = -1)} = -1$$

$$A_2 = H_a(s)(s + 2)\Big|_{(s = -2)} = 2$$

Then

$$H(z) = \frac{-1}{1 - e^{-T} z^{-1}} + \frac{2}{1 - e^{-2T} z^{-1}} = \frac{1 + z^{-1}(e^{-2T} - 2e^{-T})}{(1 - e^{-T} z^{-1})(1 - e^{-2T} z^{-1})}$$

Solution 14.5*

(a) Since $\Omega_c < \frac{\pi}{T}$ there is no aliasing introduced in obtaining the digital filter from the analog filter. Thus the frequency response $H_d(e^{j\omega})$ of

the digital filter is most easily obtained using equation 5.8. Thus

$$H_d(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} H_a\left(\frac{j\omega}{T} + j\,\frac{2\pi}{T}\,k\right)$$

and since

$$H_a(j\Omega) = 0 \qquad |\Omega| \geq \frac{\pi}{T},$$

$$H_a(e^{j\omega}) = \frac{1}{T}\,H_a\left(\frac{j\omega}{T}\right) \qquad |\omega| \leq \pi$$

$$= \frac{1}{T}\left(\frac{j\omega}{T}\right) e^{-j\frac{\omega}{T}\tau} \qquad |\omega| \leq \Omega_c T$$

$$= \frac{1}{T^2}\, j\omega\; e^{-j\frac{\omega}{T}\tau} \qquad |\omega| \leq \Omega_c T$$

We note that this has the same shape as the analog frequency response (because there is no aliasing) but with a scaling of the frequency axis.

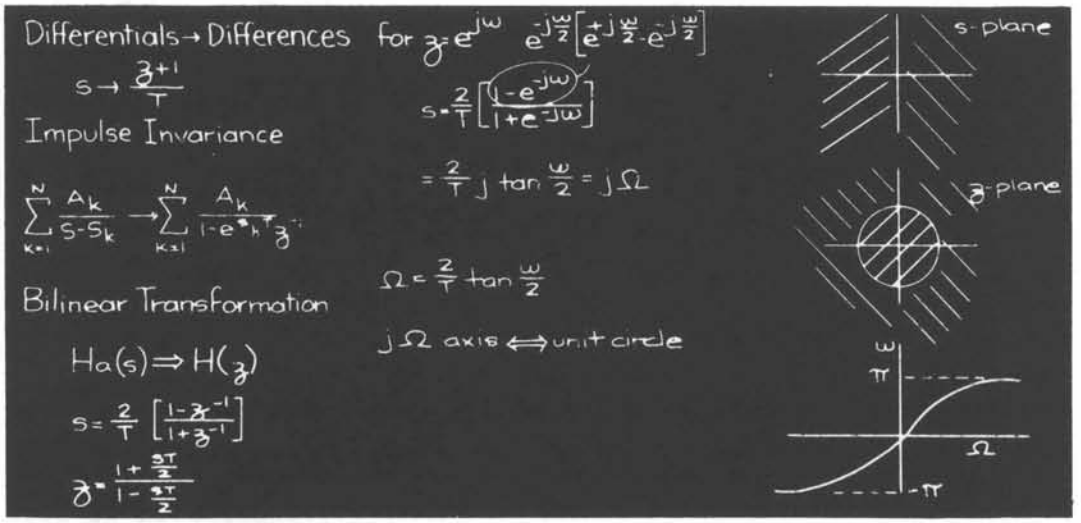
(b) If $h_d(n) = \hat{h}_d(n - n_\tau)$, $H_d(e^{j\omega})$ and $\hat{H}_d(e^{j\omega})$ are related by $H_d(e^{j\omega}) = e^{-j\omega n_\tau}\,\hat{H}_d(e^{j\omega})$

From (a) it follows, then, that $n_\tau = \frac{\tau}{T}$. For n_τ an integer, τ must be an integer multiple of the sampling period T.

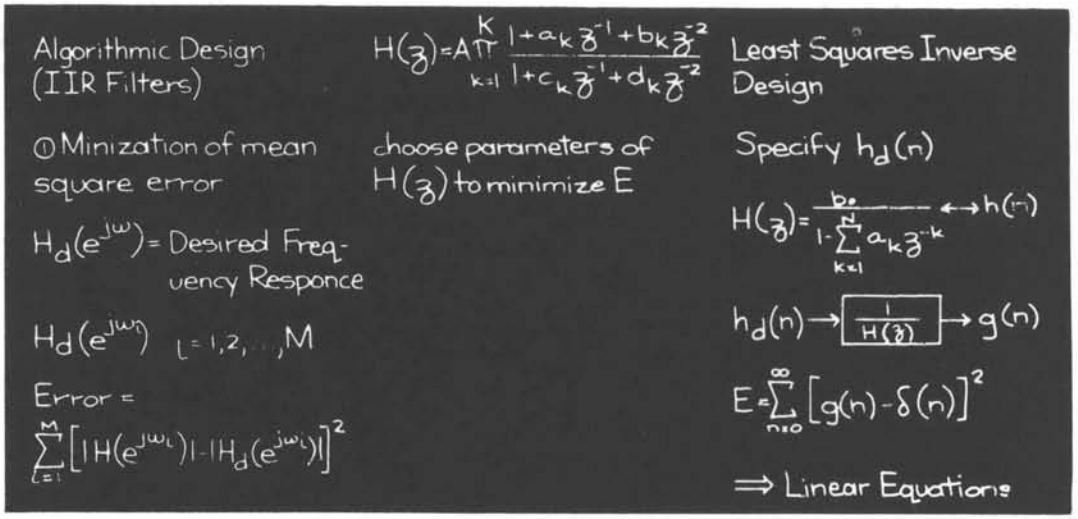
DESIGN OF IIR DIGITAL FILTERS - PART 2

1. Lesson 15 - 41 minutes

a.



b.


Algorithmic Design (IIR Filters)
$H(z) = A \prod_{k=1}^{K} \frac{1+a_k z^{-1}+b_k z^{-2}}{1+c_k z^{-1}+d_k z^{-2}}$
Least Squares Inverse Design
① Minization of mean square error
choose parameters of $H(z)$ to minimize E
Specify $h_d(n)$
$H(z) = \frac{b_0}{1-\sum_{k=1}^{N} a_k z^{-k}} \leftrightarrow h(n)$
$H_d(e^{j\omega})$ = Desired Frequency Responce
$h_d(n) \to \boxed{\frac{1}{H(z)}} \to g(n)$
$H_d(e^{j\omega_l})$ $l = 1, 2, \ldots, M$
$E = \sum_{n=0}^{\infty}[g(n)-\delta(n)]^2$
Error =
$\sum_{l=1}^{M}\left[|H(e^{j\omega_l})| - |H_d(e^{j\omega_l})|\right]^2$
$\Rightarrow$ Linear Equations

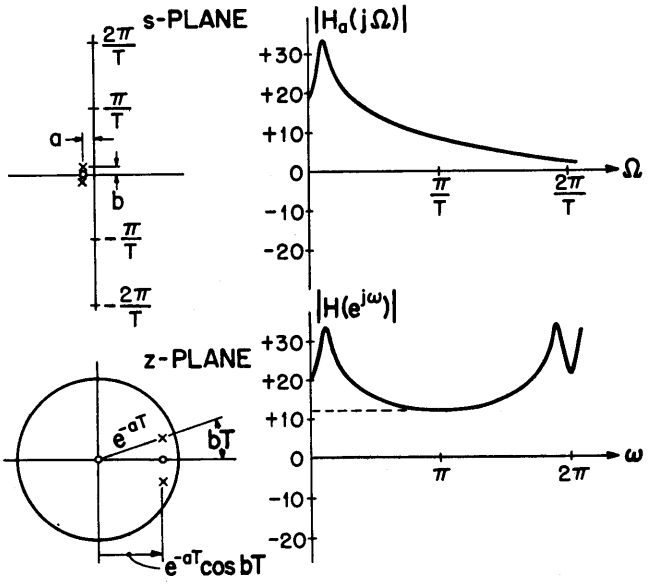
Example of impulse invariance.

$$H_a(S) = \frac{(S+a)}{(S+a)^2+b^2} = \frac{1/2}{S+a+jb} + \frac{1/2}{S+a-jb}$$

$$H(z) = \frac{1/2}{1-e^{-aT}e^{-jbT}z^{-1}} + \frac{1/2}{1-e^{-aT}e^{jbT}z^{-1}}$$

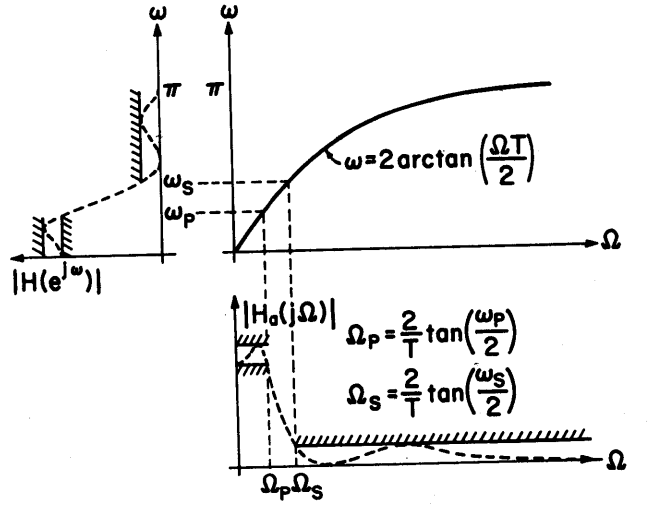
$$= \frac{1-(e^{-aT}\cos bT)\,z^{-1}}{(1-e^{-aT}e^{-jbT}z^{-1})(1-e^{-aT}e^{jbT}z^{-1})}$$

c.



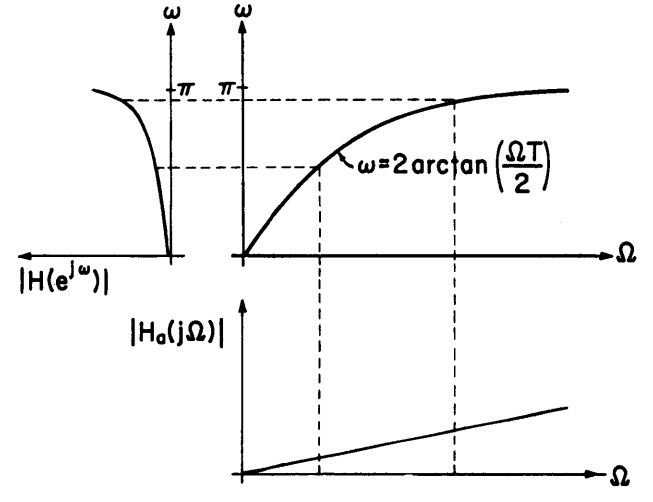
d.

Pole-zero patterns and frequency response corresponding to the example of viewgraph a.



e.

Illustration of effect of frequency warping inherent in the bilinear transformation.



f.

Illustration of effect of bilinear transformation on a piecewise constant frequency response characteristic.

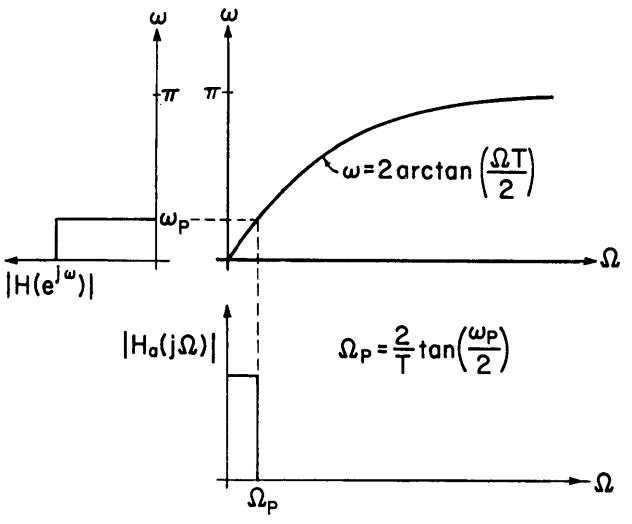
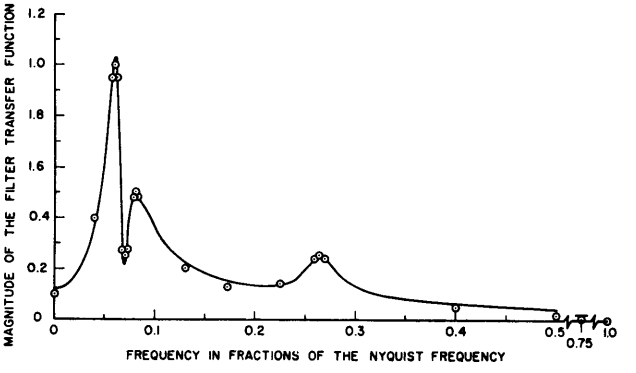


Illustration of effect of bilinear transformation on an equiripple frequency response characteristic.

g.



Example of frequency response obtained for an IIR filter designed by minimization of mean-square error.

h.

2. Correction

The first equation on board number 1 indicates that the mapping corresponding to replacing differentials by differences is $s \to \frac{z+1}{T}$. This should be corrected to $s \to \frac{z-1}{T}$.

3. Comments

In the previous lecture we introduced two techniques for transforming analog filter designs to digital filter designs. The first of these, corresponding to transforming differentials to differences is not a very useful technique for the reasons discussed. The second, that of

impulse invariance is a useful technique, although it introduces aliasing which must be accounted for.

In this lecture we begin with an illustration of impulse invariance. An important observation in this example is that the zeros of the analog transfer function don't map to the z-plane in the same way that the poles do. As we stressed in lecture 14, the impulse invariant design procedure corresponds to a mapping of the poles and residues rather than a mapping of the poles and zeros. We also observe in this example the effect of aliasing.

The second method for IIR filter design discussed in this lecture is the use of the bilinear transformation. This transformation has the advantage that it maps the entire $j\Omega$ axis in the s-plane into one revolution around the unit circle in the z-plane. Aliasing is therefore eliminated but in contrast to impulse invariance a nonlinear distortion in the frequency axis is introduced. This nonlinear distortion can be tolerated when the filter to be designed is piece-wise constant but usually cannot be foremore general filter characteristics. In the next lecture a number of these points will be illustrated through a design example.

Impulse invariance and the bilinear transformation represent the two major analytical techniques for designing digital filters through the transformation of analog designs. There are also a number of algorithmic design procedures for digital filters. We conclude this lecture with a brief introduction to two, one of which requires the solution of a set of nonlinear equations and the second of which leads to a set of linear equations.

4. Reading

Text: Review section 7.1.1 (page 407) to example 7.3 and read section 7.1.2. (page 415) to example 7.4. Also read section 7.3 (page 438).

5. Problems

Problem 15.1

In Figure P15.1-1 is shown the frequency response of a digital filter. (a) Determine and sketch the analog frequency response characteristics which, without aliasing will map to this digital frequency response when the impulse invariant transformation is applied.

(b) Sketch the analog frequency response that will map to this digital frequency response when the bilinear transformation is applied.

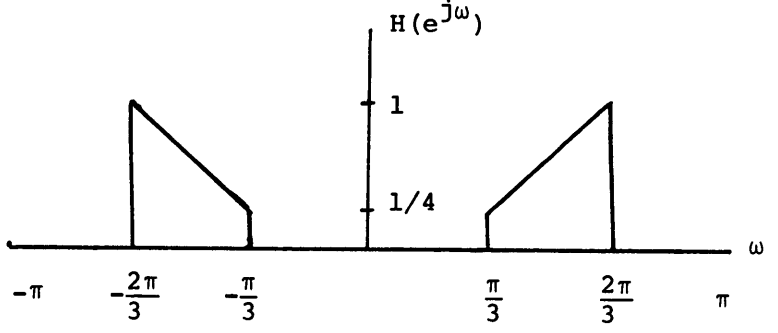


Figure P15.1-1

Problem 15.2

Repeat problem 15.1 for the digital frequency response characteristic in Figure P15.2-1.

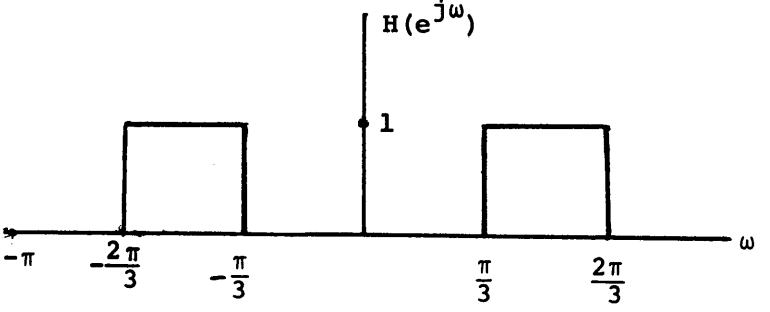


Figure P15.2-1

Problem 15.3

A digital lowpass filter is to be designed using the bilinear transformation. The specifications are that:

(i) $(.99) < |H(e^{j\omega})| \leq 1$ for $0 \leq \omega < \frac{\pi}{8}$

(ii) $|H(e^{j\omega})| < .001$ for $\frac{\pi}{6} \leq \omega < \pi$.

Determine the specifications on an analog lowpass filter which will yield the desired digital filter when the bilinear transformation is applied with T = 1.

Problem 15.4*

An analog highpass filter can be obtained from an analog lowpass filter by replacing s by 1/s in the transfer function: i.e., if $G_a(s)$ is the transfer function for a lowpass filter, then $H_a(s)$ is the transfer function for a highpass filter if

$$H_a(s) = G_a\left(\frac{1}{s}\right) .$$

Also, a digital filter can be obtained by mapping an analog filter by means of the bilinear transformation

$s = \frac{z-1}{z+1}$. [Assume for convenience that T = 2 in Eq. (5.22) of the text.]

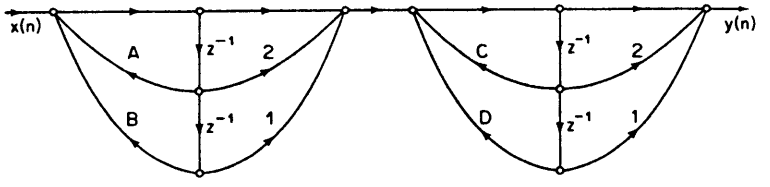


Figure P15.4-1

This mapping preserves the character of the magnitude characteristic, although the frequency scale is distorted. The network in Figure P 15.4-1 depicts a lowpass filter with cutoff frequency $\omega_L = \pi/2$.

The constants A,B,C, and D are real. Determine how to modify the coefficients in the network to obtain a highpass filter with cutoff frequency $\omega_H = \pi/2$.

Problem 15.5*

We wish to approximate a desired unit-sample response using the least-squares inverse design procedure as outlined in section 5.3.3. of text. The desired unit sample response $h_d(n)$ is given by

$$h_d(n) = \begin{cases} 1 & 0 \leq n < 10 \\ \\ 0 & \text{otherwise} \end{cases}$$

corresponding to a 9^{th} order FIR filter. This is to be approximated by a second order IIR filter of the form

$$H(z) = \frac{b_0}{1-a_1 z^{-1}-a_2 z^{-2}} .$$

Determine the coefficients b_0, a_1 and a_2.

DESIGN OF IIR DIGITAL FILTERS - PART 2

Solution 15.1

In the absence of aliasing, the transformation from analog to digital frequency response corresponding to impulse invariance is

$$H(e^{j\omega}) = \frac{1}{T} H_a\left(\frac{j\omega}{T}\right) \qquad |\omega| \leq \pi$$

Thus, in addition to a scale factor of 1/T there is a linear mapping between analog and digital frequency given by

$$\omega = \Omega T \qquad |\omega| \leq \pi.$$

The desired analog frequency response can be obtained by reflecting the digital frequency response through this transformation, as indicated in Figure S15.1-1

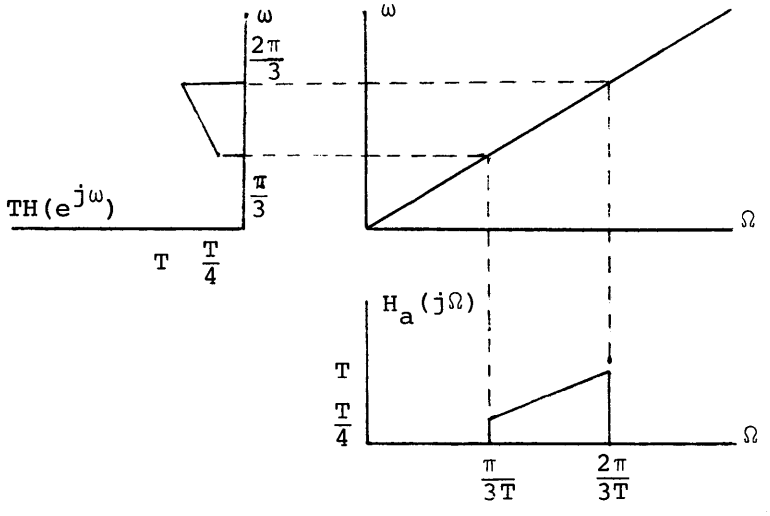


Figure S15.1-1

Note that since the frequency transformation between ω and Ω is linear, the shape of the frequency response is preserved.

(b) For the bilinear transformation the transformation between analog and digital frequency is given by

$$\omega = 2 \arctan\left(\frac{\Omega T}{2}\right) .$$

Thus, as in (a) we obtain the corresponding analog frequency response by reflecting the digital frequency response through this transformation as indicated in Figure S15.1-2.

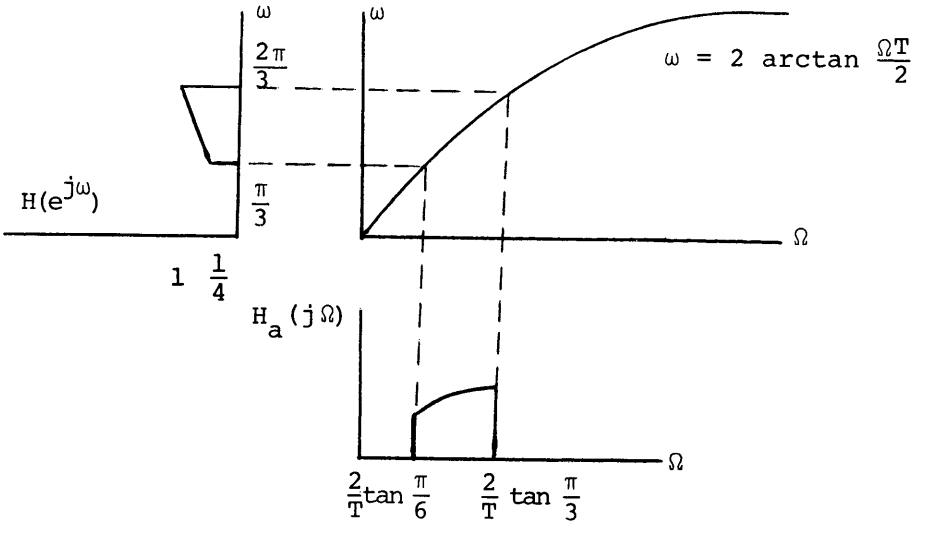


Figure S15.1-2

In this case, since the frequency transformation between ω and Ω is not linear, the linear slope of the digital frequency response does not correspond to a linear slope in the analog frequency response.

Solution 15.2

Figures S15.2-1 and S15.2-2 show the result of reflecting this frequency response characteristic through the frequency transformation for impulse invariance and the bilinear transformation respectively. In this case, since the digital frequency response is piecewise constant, its shape is preserved in the analog frequency response for both cases.

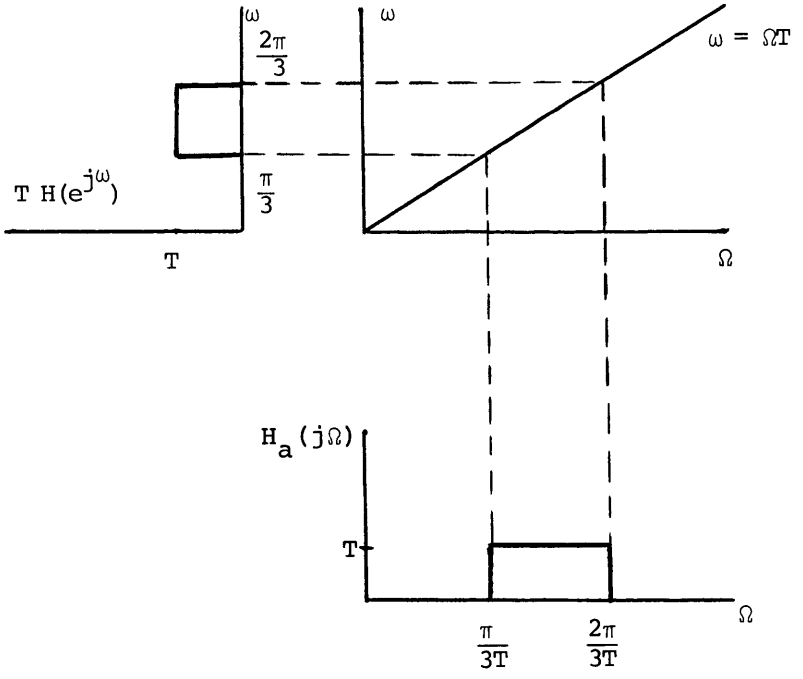


Figure S15.2-1

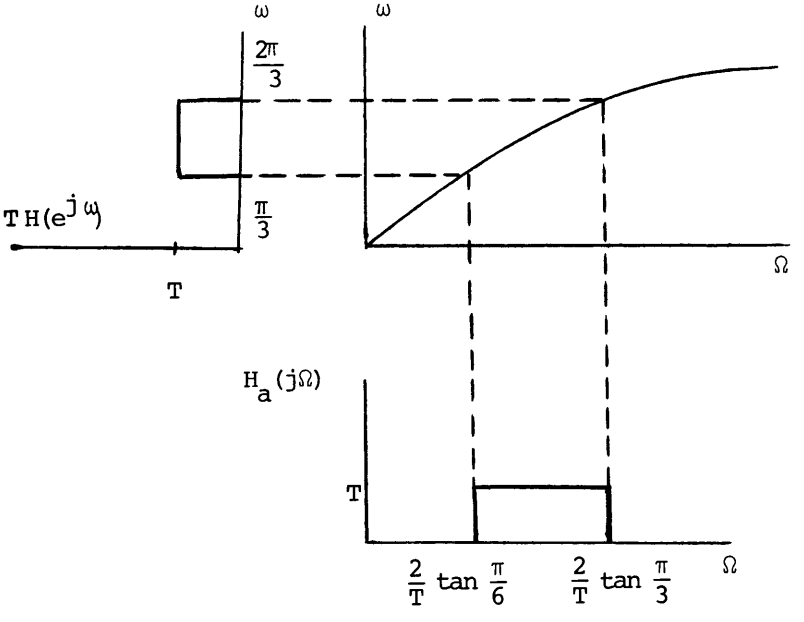


Figure S15.2-2

Solution 15.3

Since the filter is to be designed using the bilinear transformation with $T = 1$, the relation between analog and digital frequency is

$$\Omega = 2\tan\frac{\omega}{2} .$$

Thus we require that:

(i) $$(.99) < |H_a(j\Omega)| \leq 1 \text{ for } 0 \leq \Omega < 2 \tan \frac{\pi}{16}$$

and

(ii) $$|H_a(j\Omega)| < .001 \text{ for } 2\tan \frac{\pi}{12} < \Omega$$

Solution 15.4*

Since $G_a(s)$ is an analog low pass filter, $G(z)$ given by

$$G(z) = G_a[\frac{z-1}{z+1}]$$

is a digital lowpass filter. Also, since $H_a(s) = G_a(1/s)$ is an analog highpass filter,

$$H(z) = H_a[\frac{z-1}{z+1}] = G_a[\frac{z+1}{z-1}] .$$

Comparing $H(z)$ with $G(z)$ we observe that

$$H(z) = G(-z).$$

Thus, multiplying the output of each delay by -1 will convert a digital lowpass filter to a digital highpass filter. To determine the relationship between the cutoff frequencies, since $e^{j\pi}=-1$,

$$H[e^{j\omega}] = G[e^{j\pi}e^{j\omega}] = G[e^{j(\pi+\omega)}].$$

Thus the frequency response of the highpass filter is equal to that of the lowpass filter, shifted by π as illustrated in Figure S15.4-1.

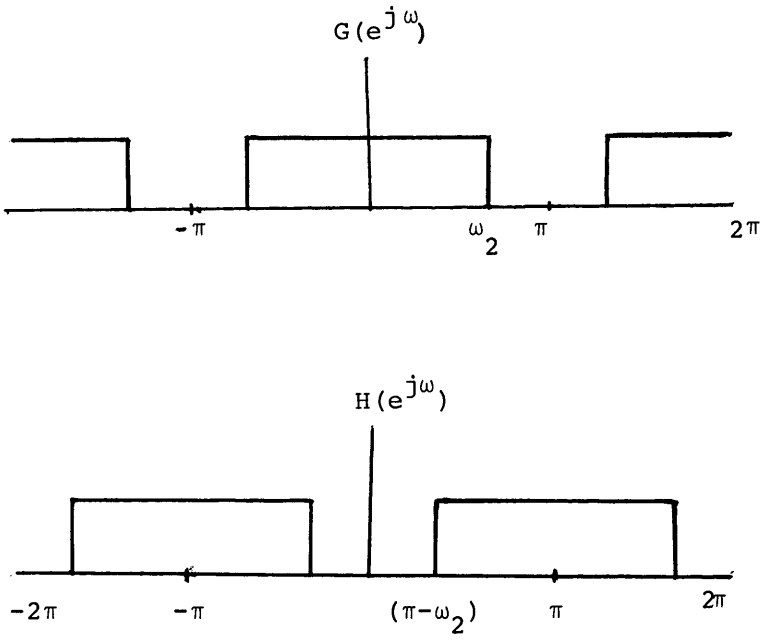


Figure S15.4-1

The cutoff frequency of the high pass filter is $(\pi-\omega_2)$ and thus with $\omega_2 = \frac{\pi}{2}$, ω_H is also $\frac{\pi}{2}$ as desired.

Finally, to determine how to modify each of the coefficients we note that multiplying the output of each delay by -1 is equivalent to changing the sign of all coefficient branches whose input has passed through an odd number of delays. Thus the coefficients A,C and the two coefficients of 2 are multiplied by -1.

Solution 15.5

The coefficient b_0 is chosen to be $h_d(0) = 1$. The coefficients a_1 and a_2 are obtained by solving the equations

$$a_1\phi(1,1) + a_2\phi(1,2) = \phi(1,0)$$

$$a_1\phi(2,1) + a_2\phi(2,2) = \phi(2,0)$$

where

$$\phi(i,r) = \sum_{n=1}^{\infty} h_d(n-r)\, h_d(n-i)$$

For the specified desired unit-sample response

$\phi(1,0) = 9$

$\phi(2,0) = 8$

$\phi(1,1) = 10$

$\phi(1,2) = 9$

$\phi(2,1) = 9$

$\phi(2,2) = 10$

Thus

$$10a_1 + 9a_2 = 9$$

$$9a_1 + 10a_2 = 8$$

Solving these equations, we obtain

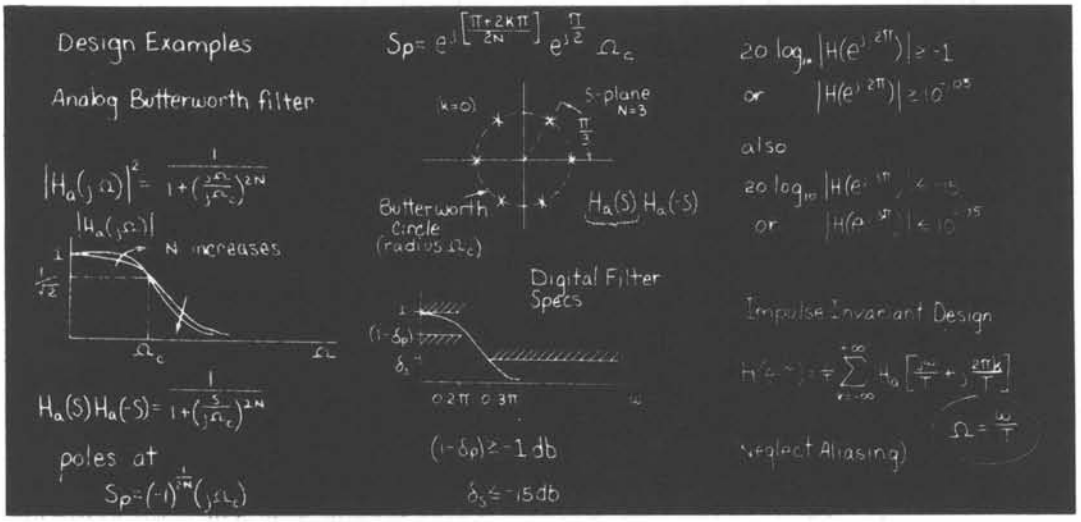
$$a_1 = \frac{18}{19}$$

$$a_2 = \frac{-1}{19} .$$

DIGITAL BUTTERWORTH FILTERS

1. Lesson 16 - 48 minutes

a.



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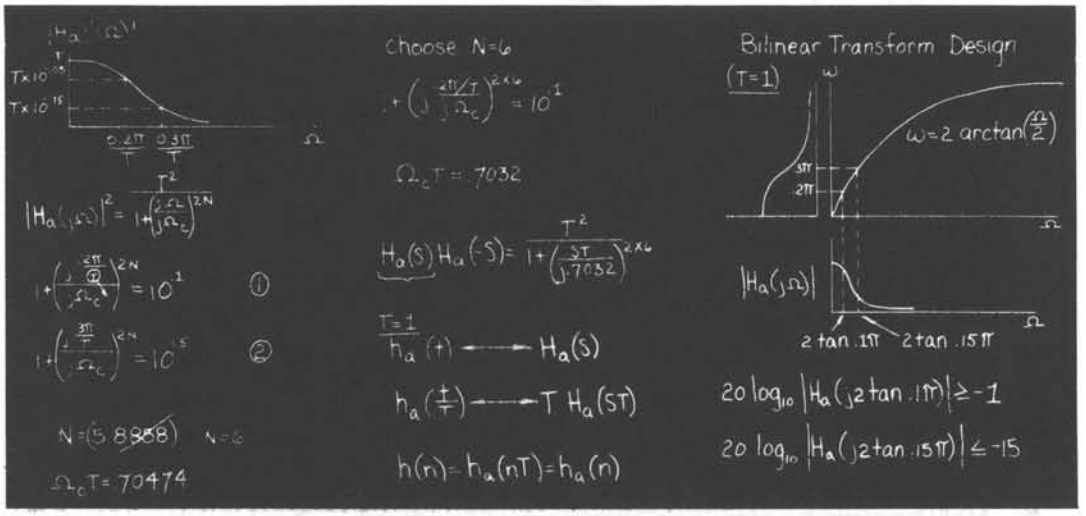
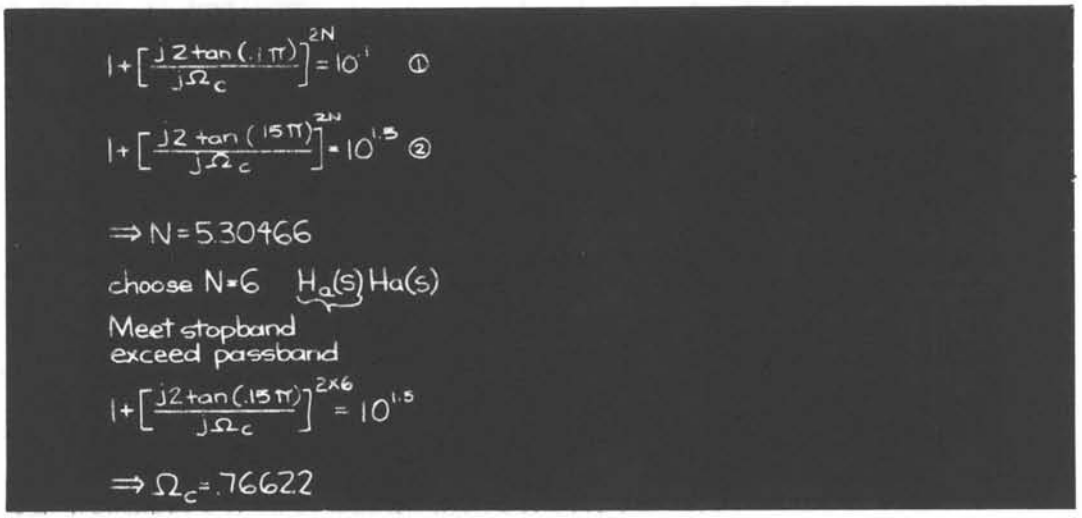
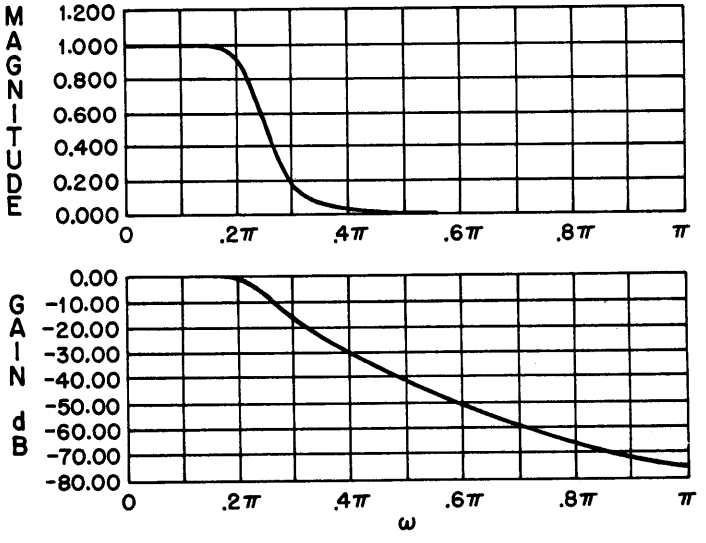


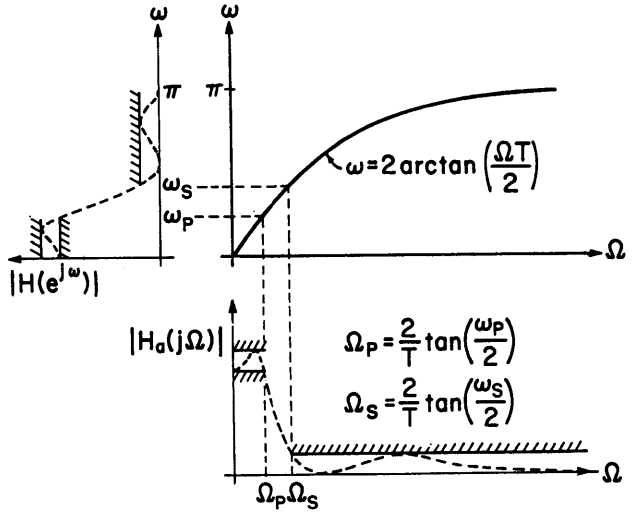
c.





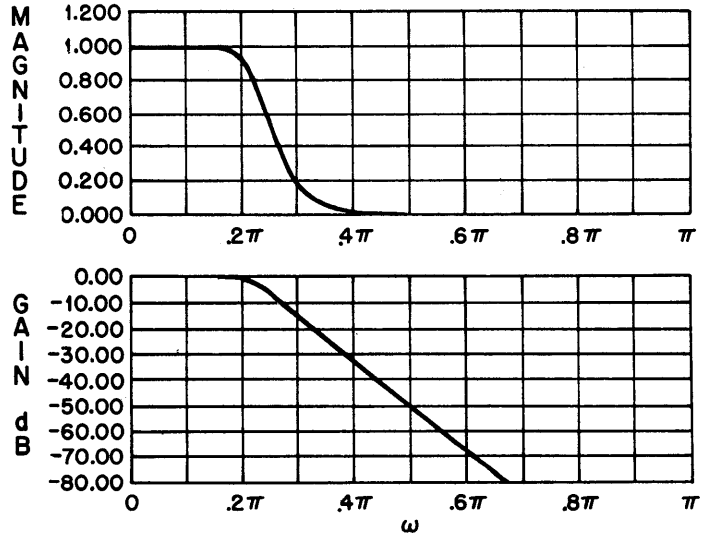
Frequency response of sixth-order digital Butterworth filter obtained by using impulse invariance.

d.



The bilinear transformation.

e.



Frequency response of sixth-order digital Butterworth filter obtained by using the bilinear transformation.

f.

2. Comments

In this lecture we discuss a design example utilizing the two techniques of impulse invariance and the bilinear transformation. We concentrate specifically on the design of a digital filter which is obtained from an analog Butterworth filter. The steps involved are to (1) transform the specifications on the digital filter to the appropriate specifications on the analog filter (2) design the analog filter based on these specifications (3) transform the analog filter to the desired digital filter. The specific example sketched out in the lecture is carried through in more detail in the text.

In both the impulse invariant and bilinear design procedures, a parameter T appears. This parameter is, in fact, unnecessary, as is illustrated in the lecture. Consequently, it is often convenient to choose this parameter to be unity.

3. Reading

Text: Appendix B1 (page 845), example 7.3 section 7.1.1 (pg. 411) and example 7.4 section 7.1.2 (page 419). You may also want to read Appendix B.2 in which Chebyshev filters are discussed, and Appendix B.3 in which elliptic filters are discussed.

4. Problems

Problem 16.1

Let $|H(j\Omega)|^2$ denote the squared magnitude function for an analog Butterworth filter of order 5 with a cutoff frequency Ω_c of $2\pi \times 10^3$. Determine and indicate in the s-plane the poles of the system function $H(s)$. Assume that the system is stable and causal.

Problem 16.2

We want to design a digital lowpass filter with a passband magnitude characteristic that is constant to within 0.75 dB for frequencies below $\omega = 0.2613\pi$ and stopband attenuation of at least 20 dB for frequencies between $\omega = 0.4018\pi$ and π. Determine the poles of the lowest order Butterworth analog transfer function which when mapped to a digital filter using impulse invariance will meet the

specifications. Indicate also how you would proceed to obtain the transfer function of the digital filter.

Problem 16.3

With the same specifications as in Problem 16.2 determine the poles of the lowest order Butterworth analog transfer function which when mapped to a digital filter using the bilinear transformation will meet the specifications. Indicate how you would proceed to obtain the transfer function of the digital filter.

Problem 16.4*

The filter designed in Problem 16.3 is to be converted to a highpass filter using a lowpass to highpass transformation as discussed in section 7.2. In the filter designed in Problem 16.3 the pass band edge occurred at $\omega = 0.2613\pi$ and the stopband edge at $\omega = 0.4018\pi$. The new highpass filter is to have the passband edge at $\omega = 0.4018\pi$. Specify how to obtain the transfer function of the highpass filter from the lowpass filter of Problem 16.3, and determine the frequency of the stopband edge.

DIGITAL BUTTERWORTH FILTERS

Solution 16.1

The squared magnitude function for a fifth order Butterworth filter with cutoff frequency $\Omega_c = 2\pi \times 10^3$ is given by

$$H(s)\,H(-s) = \frac{1}{1+\left(\frac{s}{j2\pi \times 10^3}\right)^{10}}$$

The poles of $H(s)\,H(-s)$ are the roots of $1 + \left(\frac{s}{j2\pi \times 10^3}\right)^{10} = 0$ or

$$s = (-1)^{\frac{1}{10}} (j2\pi \times 10^3)$$

as indicated in Figure S16.1-1

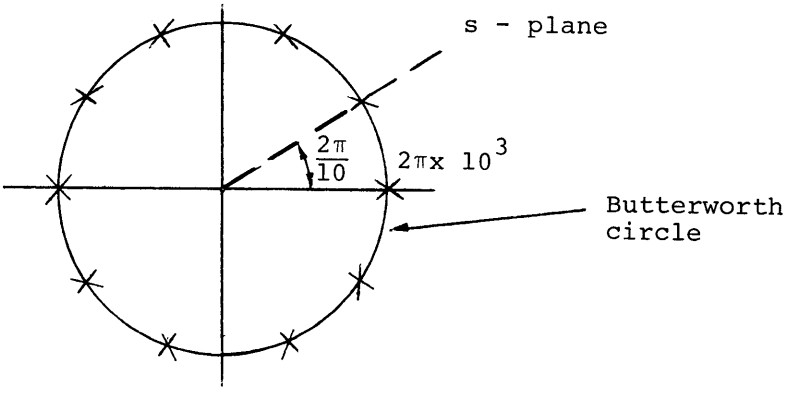


Figure S16.1-1

Since H(s) corresponds to a stable, causal filter, we factor the squared magnitude function so that the left-half plane poles correspond to H(s) and the right-half plane poles correspond to H(-s). Thus the poles of H(s) are as indicated in Figure S16.1-2.

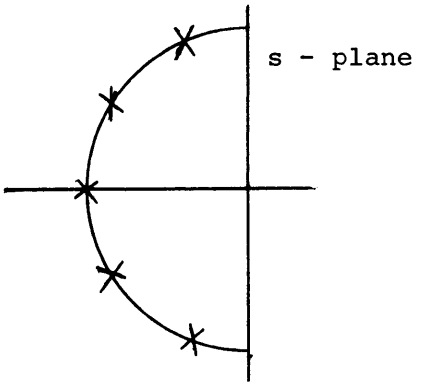


Figure S16.1-2

Solution 16.2

Since the Butterworth filter has a monotonic frequency response with unity magnitude at $\omega = 0$ the stated specifications will be met if we require that

$$|H(e^{j0})| = 1$$

$$-0.75 \leq 20 \log_{10}|H(e^{j0.2613\pi})|$$

$$20 \log_{10}|H(e^{j0.4018\pi})| \leq -20$$

or, equivalently,

$$|H(e^{j0.2613\pi})|^2 \geq 10^{-.075}$$

and

$$|H(e^{j0.4018\pi})|^2 \leq 10^{-2}$$

Using impulse invariance with $T = 1$ and neglecting aliasing we require that the analog filter $H_a(j\Omega)$ meet the specifications

$$|H_a(j0.2613\pi)|^2 \geq 10^{-.075}$$

$$|H_a(j0.4018\pi)|^2 \leq 10^{-2}$$

We will first consider meeting these specifications with equality. Thus,

$$1 + \left(\frac{j0.2613\pi}{j\Omega_c}\right)^{2N} = 10^{.075}$$

$$1 + \left(\frac{j0.4018\pi}{j\Omega_c}\right)^{2N} = 10^{2}$$

or

$$2N \log(0.4018\pi) - 2N \log\Omega_c = \log(10^2-1)$$

$$2N \log(0.2613\pi) - 2N \log\Omega_c = \log(10^{.075}-1)$$

Subtracting we obtain

$$2N \log\left(\frac{0.4018\pi}{0.2613\pi}\right) = \log\left(\frac{10^2-1}{10^{.075}-1}\right)$$

or

$$N = 7.278.$$

Since N must be an integer, we choose N = 8. Then, to compensate for the effect of aliasing we can choose Ω_c to meet the passband edge specifications in which case the stopband specifications will be exceeded. Determining Ω_c on this basis we have

$$\log\Omega_c = \log(0.2613\pi) - \frac{1}{2N}\log(10^{.075}-1)$$

or

$$\Omega_c = 0.911.$$

Thus the analog filter squared magnitude function is

$$H_a(s)\,H_a(-s) = \frac{1}{1 + \left(\frac{s}{j0.911}\right)^{16}} .$$

The poles of the squared magnitude function are indicated in Figure S16.2-1.

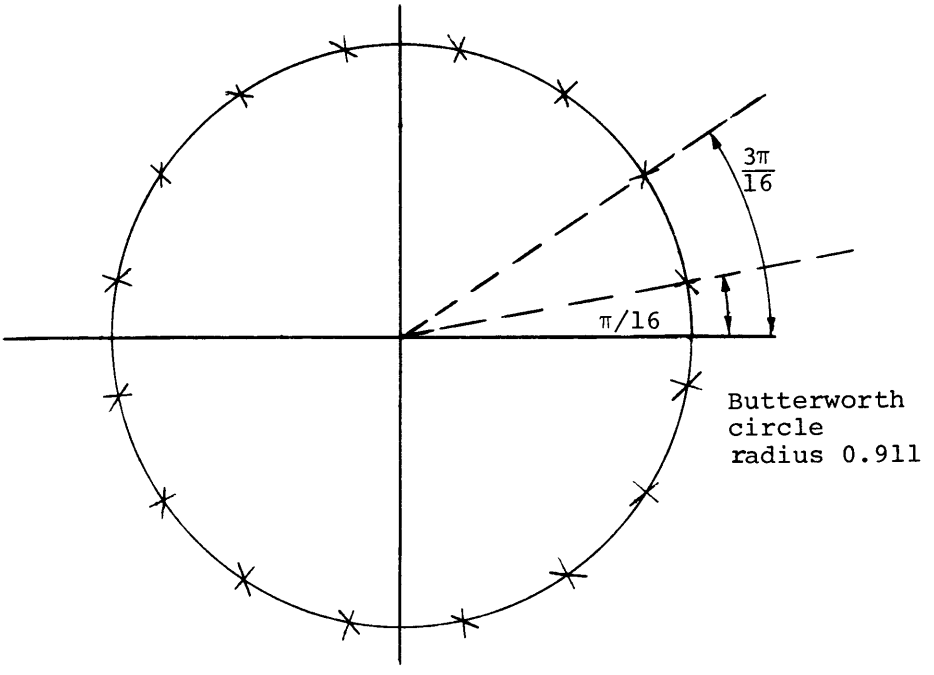


Figure S16.2-1

Therefore $H_a(s)$ has four complex conjugate pole pairs as indicated below:

pole-pair 1: $0.911\, e^{j\frac{9\pi}{16}}$, $0.911 e^{-j\frac{9\pi}{16}}$

pole-pair 2: $0.911\, e^{j\frac{11\pi}{16}}$, $0.911 e^{-j\frac{11\pi}{16}}$

pole-pair 3: $0.911\, e^{j\frac{13\pi}{16}}$, $0.911\, e^{-j\frac{13\pi}{16}}$

pole-pair 4: $0.911\, e^{j\frac{15\pi}{16}}$, $0.911\, e^{-j\frac{15\pi}{16}}$

From these pole-pairs it is straightforward to express $H_a(s)$ in factored form as

$$H_a(s) = \prod_{k=1}^{4} \frac{A}{(s-s_k)(s-s_k^*)}$$

where the factor A is determined so that $H_a(s)$ has unity gain at zero frequency, i.e.

$$H_a(0) = 1 \text{ or } A = \prod_{k=1}^{4} |s_k|^2 = (0.911)^8.$$

To transform this analog filter to the desired digital filter using impulse invariance, we would first expand $H_a(s)$ in a partial fraction expansion as

$$H_a(s) = \sum_{k=1}^{4} \left[\frac{a_k}{s-s_k} + \frac{a_k^*}{s-s_k^*} \right] .$$

The desired digital filter transfer function is then

$$H(z) = \sum_{k=1}^{4} \left[\frac{a_k}{1-e^{s_k} z^{-1}} + \frac{a_k^*}{1-e^{s_k^*} z^{-1}} \right] .$$

The residues a_k and a_k^* are evaluated as:

$$a_k = H_a(s) \ (s-s_k) \Big|_{s=s_k} .$$

Solution 16.3

Again the specifications on the digital filter are that

$$|H(e^{j0.2613\pi})|^2 \geq 10^{-.075}$$

and

$$|H(e^{j0.4018\pi})|^2 \leq 10^{-2} .$$

To obtain the specifications on the analog filter we must determine the analog frequencies Ω_p and Ω_s which will map to the digital frequencies of 0.2613π and 0.4018π respectively when the bilinear transformation is applied. With T = 1 in the bilinear transformation, these are given by

$$\Omega_p = 2 \tan \left(\frac{.2613\pi}{2} \right) = .8703$$

$$\Omega = 2 \tan \left(\frac{.4018\pi}{2} \right) = 1.4617$$

Thus, the specifications on the analog Butterworth filter are:

$$|H_a(j.8703)|^2 \geq 10^{-.075}$$

$$|H_a(j1.4617)|^2 \leq 10^{-2}$$

As in problem 16.2 we will consider meeting these specifications with equality. Thus:

$$1 + \left(\frac{.8703}{\Omega_c}\right)^{2N} = 10^{.075}$$

$$1 + \left(\frac{1.4617}{\Omega_c}\right)^{2N} = 10^{2}$$

Solving for N we obtain

$$N = 6.04.$$

This is so close to 6 that we might be willing to relax the specifications slightly and use a 6^{th} order filter. Alternatively we would use a 7^{th} order filter and exceed the specifications. Choosing the latter and picking Ω_c to exactly meet the pass band specifications,

$$14\ [\log(.8703) - \log\Omega_c] = \log[10^{.075}-1]$$

$$\Omega_c = .9805\ .$$

Thus the analog filter squared magnitude function is

$$H_a(s)\ H_a(-s) = \frac{1}{1 + \left(\frac{s}{.9805}\right)^{14}}$$

the poles of which are indicated in Figure S16.3-1.

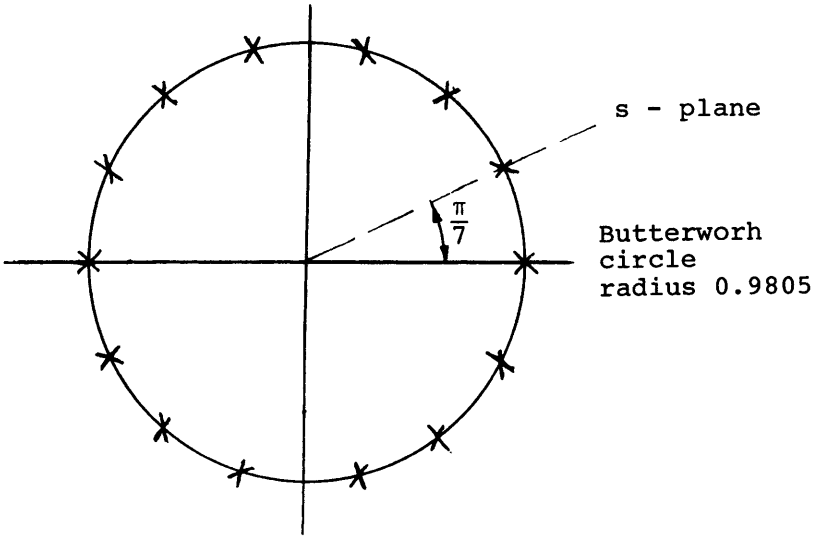


Figure S16.3-1

Therefore, $H_a(s)$ has three complex conjugate pole-pairs and one real pole as indicated below:

real pole: $-.9805$

pole-pair 1: $.9805\, e^{j\frac{8\pi}{14}}$, $.9805\, e^{-j\frac{8\pi}{14}}$

pole-pair 2: $.9805\, e^{j\frac{10\pi}{14}}$, $.9805\, e^{-j\frac{10\pi}{14}}$

pole-pair 3: $.9805\, e^{j\frac{12\pi}{14}}$, $.9805\, e^{-j\frac{12\pi}{14}}$

From these pole locations $H_a(s)$ can be easily expressed in factored form. The digital filter transfer function is then obtained as:

$$H(z) = H_a(s)\Big|_{s=2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}.$$

Solution 16.4

Let $H_\ell(z)$ denote the transfer function of the lowpass filter designed in Problem 16.3 and $H_h(z)$ the transfer function of the desired high-pass filter. To obtain $H_\ell(z)$ from $H_h(z)$ we apply the lowpass to high pass transformation (see table 7.1 page 434 of the text).

$$z^{-1} = - \frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$$

where

$$\alpha = - \frac{\cos\left(\frac{.4018\pi + .2613\pi}{2}\right)}{\cos\left(\frac{.4018\pi - .2613\pi}{2}\right)}$$

$$\alpha = - 0.517$$

$$H_h(z) = H_\ell \left[- \frac{1+\alpha z^{-1}}{z^{-1}+\alpha} \right]$$

Next, let θ_s denote the stopband edge frequency for the lowpass filter and ω_s the stopband edge frequency for the highpass filter. Then, inverting the transformation

$$z^{-1} = - \frac{z^{-1} + \alpha}{1+\alpha z^{-1}}$$

we have

$$z^{-1} = - \left(\frac{z^{-1} + \alpha}{\alpha z^{-1} + 1} \right)$$

Thus,

$$e^{-j\omega_s} = - \left(\frac{e^{-j\theta_s} + \alpha}{\alpha e^{-j\theta_s} + 1} \right)$$

then

$$\omega_s = \tan^{-1} \left(\frac{(1-\alpha^2) \sin\theta_s}{-2\alpha - (1+\alpha^2) \cos\theta_s} \right)$$

From this we obtain

$$\omega_s = .2616 \ \pi \ .$$

DESIGN OF FIR DIGITAL FILTERS

1. Lesson 17 -39 minutes

DESIGN OF FIR DIGITAL FILTERS

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

FOR LINEAR PHASE $h(n) = h(N-1-n)$

BASIC DESIGN METHODS:

① WINDOWS

② FREQUENCY SAMPLING

③ EQUIRIPPLE DESIGN

Basic design methods for FIR digital filters

a.

DESIGN OF FIR FILTERS USING WINDOWS

DESIRED UNIT SAMPLE RESPONSE: $h_d(n)$

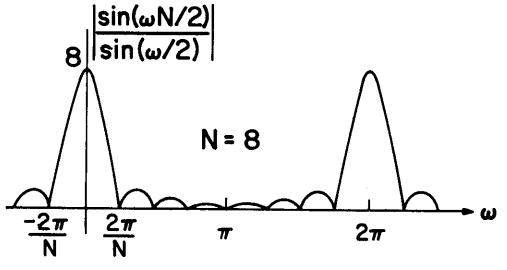
$$h(n) = w(n)\, h_d(n)$$

$$w(n) = 0 \qquad n < 0,\ n > (N-1)$$

$$H(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\theta})\, W[e^{j(\omega-\theta)}]\, d\theta$$

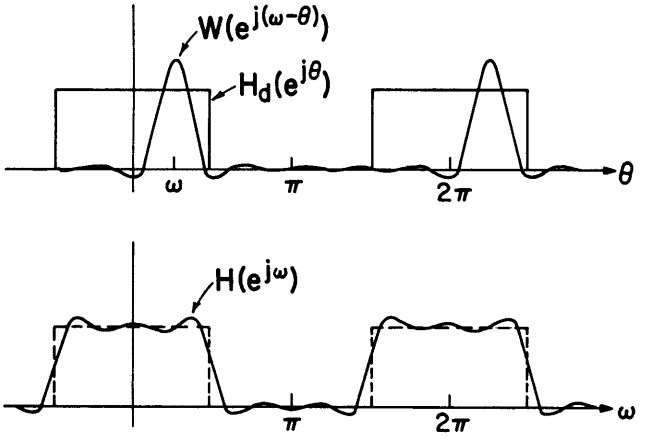
Design of FIR filters using the window method.

b.



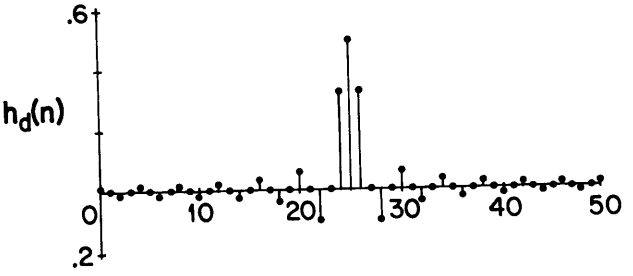
Magnitude of the Fourier transform for an eight point rectangular window.

c.



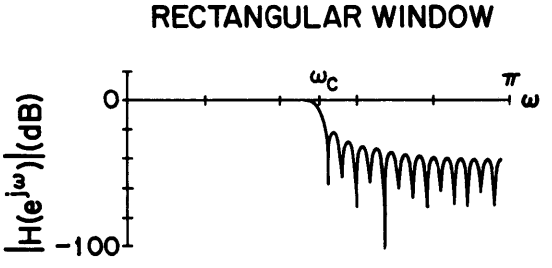
Effect of convolving the Fourier transform of a rectangular window with an ideal low pass filter characteristic.

d.



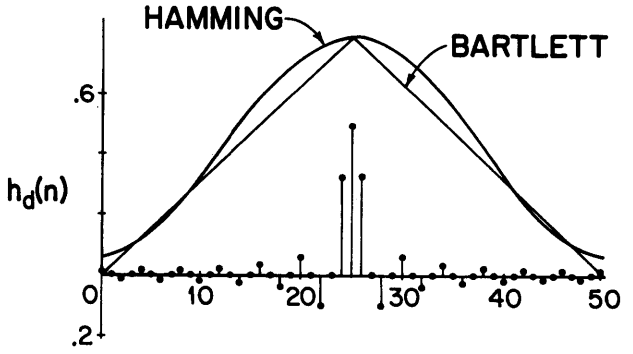
Unit-sample response of an ideal low-pass filter truncated by a 51-point rectangular window.

e.



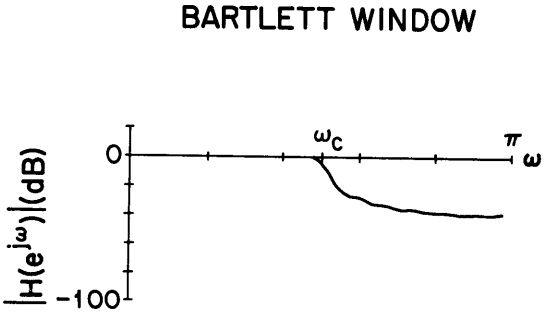
Frequency response corresponding to the unit-sample response in viewgraph e.

f.



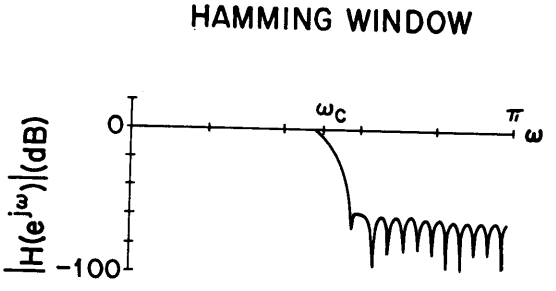
The Hamming and Bartlett windows.

g.



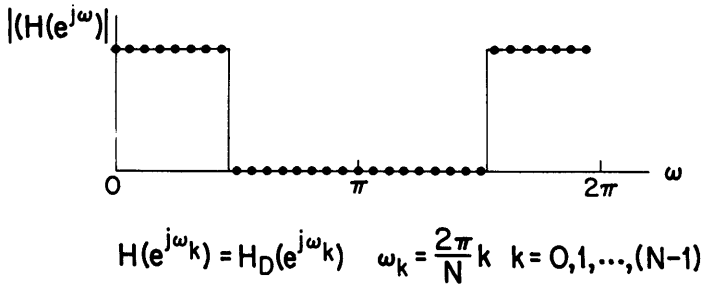
Frequency response of an FIR lowpass filter obtained by multiplying the unit-sample response of an ideal low pass filter by a Bartlett window.

h.



Frequency response of an FIR lowpass filter filter obtained by multiplying the unit sample response of an ideal lowpass filter by a Hamming window. (Note that the stopband attenuation is approximately 65 db not 30 db as stated in the lecture.)

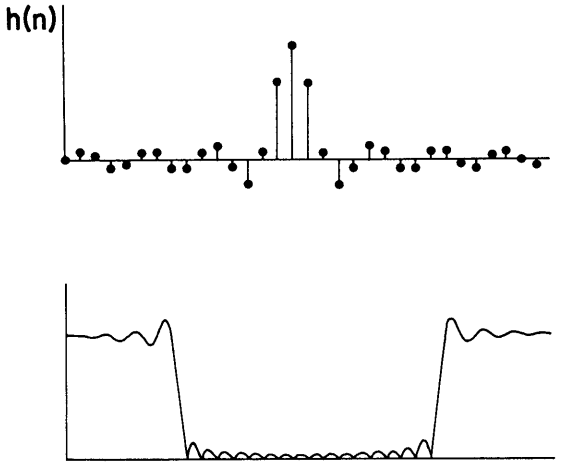
i.



$$H(e^{j\omega_k}) = H_D(e^{j\omega_k}) \quad \omega_k = \frac{2\pi}{N}k \quad k = 0,1,\cdots,(N-1)$$

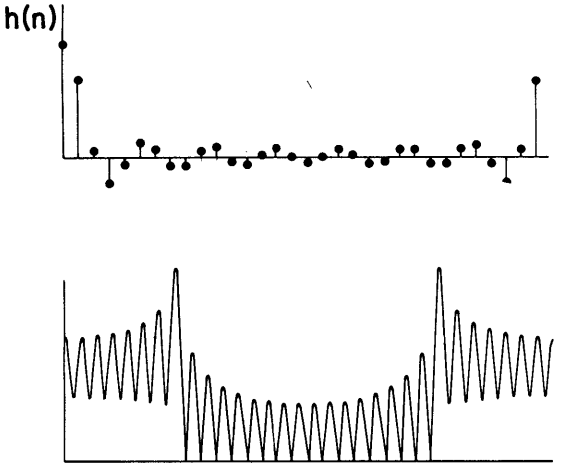
Equally spaced frequency samples of an ideal low pass filter.

j.



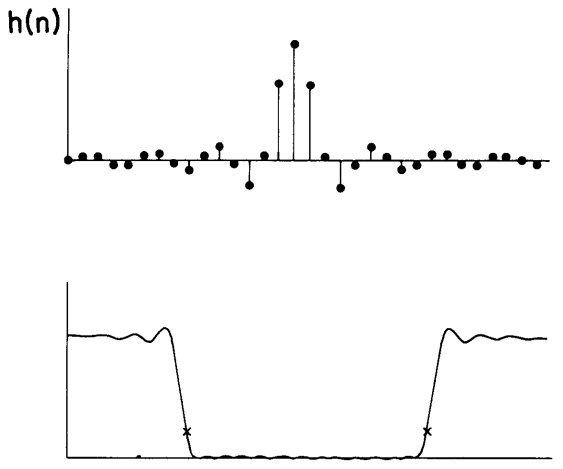
A unit-sample response the magnitude of whose DFT is equal to the frequency samples in viewgraph j.. The bottom trace is the magnitude of the Fourier transform of this unit-sample response.

k.



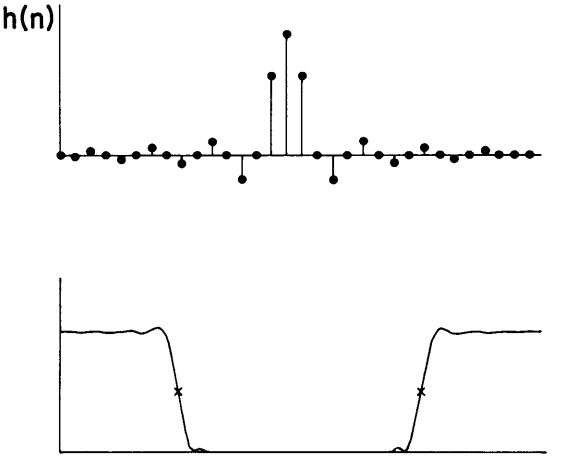
Another unit-sample response the magnitude of whose DFT is equal to the frequency samples in viewgraph j.. The bottom trace is the magnitude of the Fourier transform of this unit-sample response.

l.



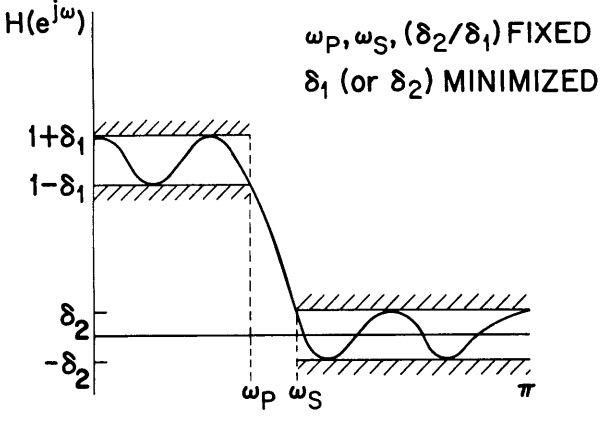
Unit sample response and frequency response when one frequency sample is moved from the stopband into the transition band.

m.



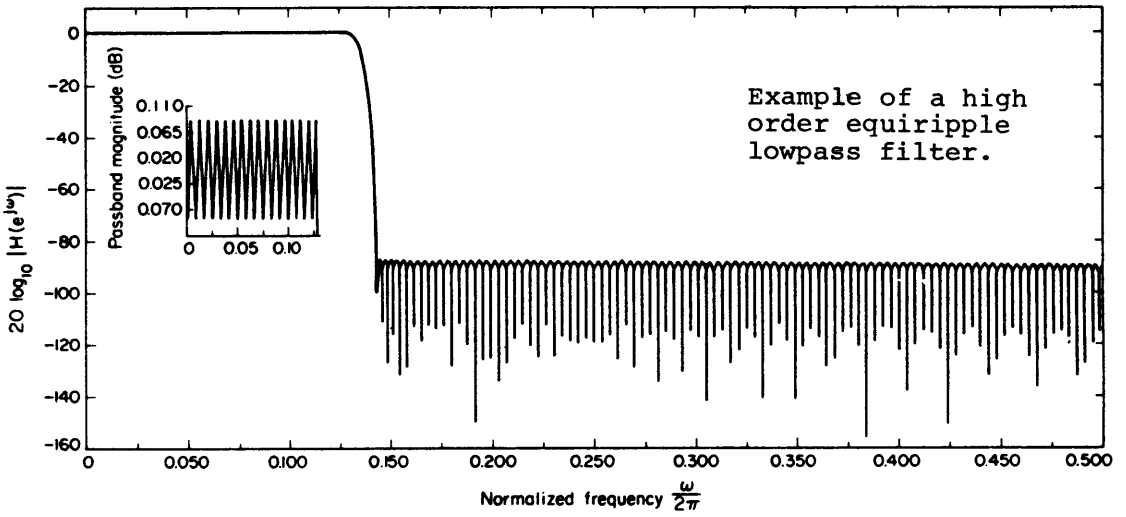
Similar to viewgraph m. with a different value of the frequency sample in the transition band.

n.



Equiripple approximation of a lowpass filter.

o.



p.

2. Comments

FIR filters are an important class of digital filters, and in contrast with continuous-time FIR filters, the implementation of digital filters of this type is relatively straightforward.

Design techniques for FIR digital filters are generally carried out directly in the discrete-time domain. This lecture introduces the three primary design techniques, specifically the window method, the frequency sampling method, and the algorithmic design of optimum filters. The window method basically begins with a desired unit-sample response which is then truncated by means of a finite duration window. In the frequency sampling method, the frequency response of the FIR filter is specfied in terms of samples of the desired frequency response. The first two design procedures generally do not result in optimum filter designs. There is available an algorithmic design procedure which generates optimum equiripple FIR filter designs. While I mention this technique only briefly in the lecture, it is developed in considerable detail in the text in sections 7.6 and 7.7

3. Reading

Text: Section 7.4 (page 444) and section 7.8.

4. Problems

Problem 17.1

In the window method of design for lowpass FIR filters we indicated that the transition width of the resulting filter is dependent primarily on the width of the main lobe of the Fourier transform of the window. For the purposes of this problem, we define the main lobe as the symmetric interval between the first negative and positive frequencies at which $W(e^{j\omega}) = 0$. Consider the following three windows:

Rectangular: $w_R(n) = 1$ $\quad |n| \leq N - 1$

$= 0$ $\quad$ otherwise

Bartlett: $w_B(n) = 1 - \frac{|n|}{N}$ $\quad |n| \leq N - 1$

$= 0$ $\quad$ otherwise

Raised cosine: $w_H(n) = \alpha + \beta\cos(\pi n/(N-1))$ $\quad |n| \leq N - 1$

$= 0$ $\quad$ otherwise

(If $\alpha=\beta=0.5$ this is the Hanning window and if $\alpha=0.54$ and $\beta=0.46$ this is the Hamming window.)

Determine the Fourier transforms of each of these windows. Also, determine the width of the main lobe for each window, assuming that $N >> 1$.

Problem 17.2

Let $h_1(n)$ and $h_2(n)$ denote the unit sample responses of two FIR filters of length 8. The two are related by a four-point circular shift, i.e.,

$$h_2(n) = h_1((n-4))_8 \, R_8(n) \ .$$

The Fourier transform of $h_1(n)$ is sketched in Figure P17.2-1, and corresponds to a low pass filter with cut-off frequency $\frac{\pi}{2}$.

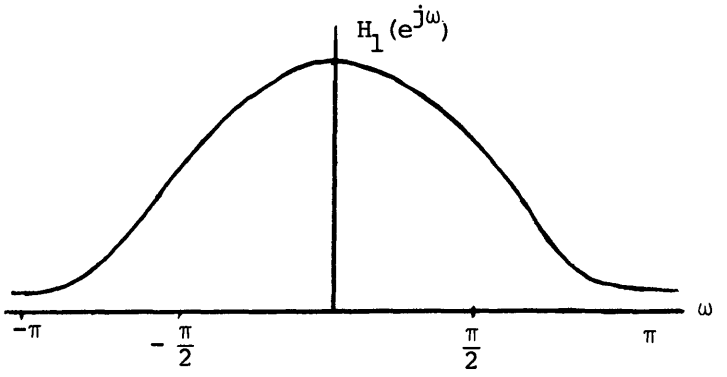


Figure P17.2-1

(a) Determine the relationship between the DFT of $h_1(n)$ and $h_2(n)$ and show in particular that the magnitude of the DFT of $h_1(n)$ and $h_2(n)$ are equal.

(b) Would it also be reasonable to consider $h_2(n)$ to correspond to a lowpass filter with cut-off frequency of $\frac{\pi}{2}$?

Problem 17.3

Let $h_d(n)$ denote the unit-sample response of an ideal desired system with frequency response $H_d(e^{j\omega})$ and let $h(n)$ denote the unit-sample response, of the length N sampled, for an FIR system with frequency response $H(e^{j\omega})$. In Sec. 5.6 it was asserted that a rectangular window of length N samples applied to $h_d(n)$ will produce a unit-sample response $H(n)$ such that the mean-square error

$$\varepsilon^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_d(e^{j\omega}) - H(e^{j\omega})|^2 \, d\omega$$

is minimized.

(a) The error function $E(e^{j\omega}) = H_d(e^{j\omega}) - H(e^{j\omega})$ can be expressed as the power series

$$E(e^{j\omega}) = \sum_{n=-\infty}^{\infty} e(n)e^{-j\omega n}$$

Find the coefficients $e(n)$ in terms of $h_d(n)$ and $h(n)$.

(b) Express the mean-square error ε^2 in terms of the coefficients $e(n)$.

(c) Show that for a unit-sample response $h(n)$ of length N samples, ε^2 is minimized when

$$h(n) = \begin{cases} h_d(n), & 0 \leq n \leq N - 1 \\ 0, & \text{otherwise} \end{cases}$$

That is, the rectangular window gives the best mean-square approximation to a desired frequency response for a fixed value of N.

DESIGN OF FIR DIGITAL FILTERS

Solution 17.1

Rectangular window:

$$W_R(e^{j\omega}) = \sum_{n=-N+1}^{N-1} e^{-j\omega n}$$

$$= \sum_{n=0}^{2N-2} e^{-j\omega(n-N+1)}$$

$$= e^{-j\omega(-N+1)} \sum_{n=0}^{2N-2} e^{-j\omega n}$$

$$= e^{-j\omega(-N+1)} \quad \frac{1-e^{-j\omega(2N-1)}}{1-e^{-j\omega}}$$

$$W_R(e^{j\omega}) = \frac{\sin\left(\left(\frac{2N-1}{2}\right)\omega\right)}{\sin\frac{\omega}{2}}$$

The width of the main lobe is $\frac{4\pi}{2N-1}$ which, for $N >> 1$ is approximately $\frac{2\pi}{N}$.

Bartlett window:

Define $\tilde{w}_R(n) = 1 \qquad 0 \leq n \leq N-1$

$= 0 \qquad$ otherwise

Then $\tilde{w}_B(n-N+1) = \frac{1}{N}\ \tilde{w}_R(n) * \tilde{w}_R(n)$

From Eq. 7.76 of the text (with N = M + 1).

$$\tilde{W}_R(e^{j\omega}) = e^{-j\frac{\omega}{2}(N-1)} \frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}$$

Therefore

$$W_B(e^{j\omega}) = \frac{1}{N}\left(\frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}\right)^2$$

In this case the width of the main lobe is $\frac{4\pi}{N}$ which is twice that for the rectangular window.

Raised cosine window:

$$W_H(e^{j\omega}) = \sum_{-(N-1)}^{N-1} \left[\alpha + \beta\cos\left(\frac{\pi n}{N-1}\right)\right] e^{-j\omega n}$$

$$= \sum_{-(N-1)}^{N-1} \alpha e^{-j\omega n} + \frac{\beta}{2} \sum_{-(N-1)}^{N-1} e^{-jn\left(\omega+\frac{\pi}{N-1}\right)} + \sum_{-(N-1)}^{N-1} e^{-jn\left(\omega-\frac{\pi}{N-1}\right)}$$

$$= \alpha \frac{\sin\left(\omega\left(\frac{2N-1}{2}\right)\right)}{\sin\frac{\omega}{2}} + \frac{\beta}{2}\left[\frac{\sin\left[\left(\omega+\frac{\pi}{N-1}\right)\frac{2N-1}{2}\right]}{\sin\left[\frac{1}{2}\left(\omega+\frac{\pi}{N-1}\right)\right]} + \frac{\sin\left[\left(\omega-\frac{\pi}{N-1}\right)\frac{2N-1}{2}\right]}{\sin\left[\frac{1}{2}\left(\omega-\frac{\pi}{N-1}\right)\right]}\right]$$

Assuming that N >> 1, this can be rewritten as

$$W_H(e^{j\omega}) \approx \alpha \frac{\sin\omega N}{\sin\frac{\omega}{2}} + \frac{\beta}{2}\left[\frac{\sin(N\omega+\pi)}{\sin\frac{1}{2N}(N\omega+\pi)} + \frac{\sin(N\omega-\pi)}{\sin\frac{1}{2N}(N\omega-\pi)}\right]$$

This is the superposition of three terms as sketched in Figure S17.1-1.

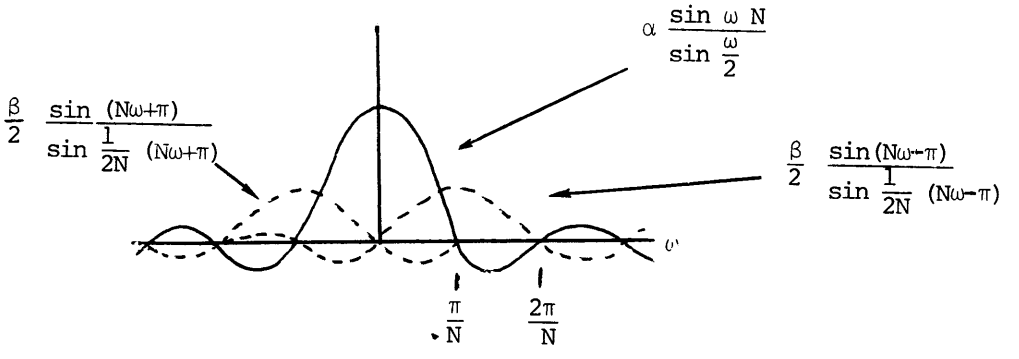


Figure S17.1-1

From this figure we observe that the first values of ω for which the superposition will be zero are $\omega = \pm \frac{2\pi}{N}$. Consequently for this window also, the width of the main lobe is $\frac{4\pi}{N}$.

Solution 17.2

(a) Since $h_1(n)$ and $h_2(n)$ are related by a circular shift, their DFTs are related by

$$H_2(k) = W_8^{4k} H_1(k) = (-1)^k H_1(k) .$$

Thus, their magnitudes are equal.

(b) Since the DFT corresponds to samples of the Fourier transform, the values of $H_1(k)$ are the samples of $H_1(e^{j\omega})$ indicated in Figure S17.2-1

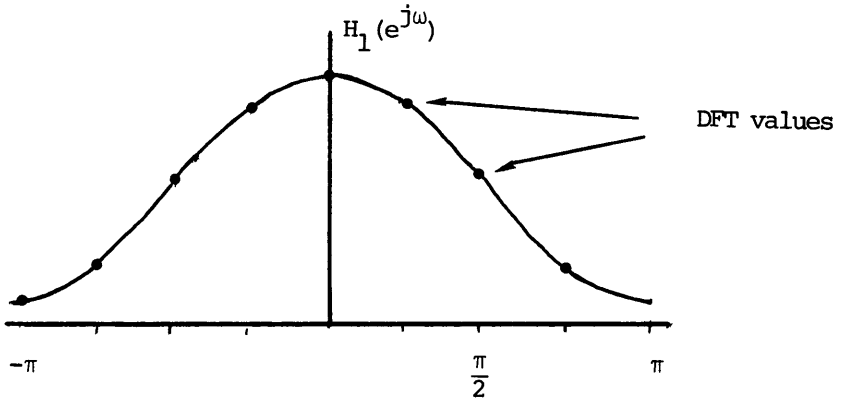


Figure S17.2-1

From (a), $H_2(k) = (-1)^k H_1(k)$. Thus the values $H_2(k)$ are as indicated in Figure S17.2-2. Since these alternate in polarity, the continuous frequency response of which these are samples, must go through zero in between these samples as indicated.

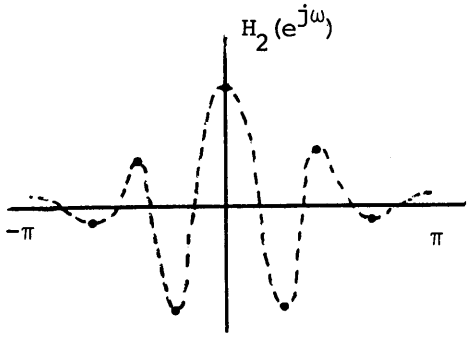


Figure S17.2-2

Thus $h_2(n)$ obviously does not correspond to a good low pass filter, even though the magnitude of its DFT values are identical to those of the low pass filter $h_1(n)$.

Solution 17.3

(a) Since $E(e^{j\omega}) = H_d(e^{j\omega}) - H(e^{j\omega})$, and $e(n)$ is the inverse Fourier transform of $E(e^{j\omega})$,

$$e(n) = h_d(n) - h(n).$$

(b) $$\varepsilon^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |E(e^{j\omega})|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{n=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} e(n)e(k)e^{-j\omega n}e^{j\omega k}\right] d\omega \quad .$$

Interchanging the order of integration and summation

$$\varepsilon^2 = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} e(n)e(k)\underbrace{\int_{-\pi}^{\pi} e^{j\omega(k-n)} d\omega}_{\begin{array}{l}=0 \quad k \neq n \\ =2\pi \quad k = n\end{array}}$$

Thus,

$$\varepsilon^2 = \sum_{n=-\infty}^{+\infty} e^2(n) \ .$$

(c) From the results of parts (a) and (b),

$$\varepsilon^2 = \sum_{n=-\infty}^{+\infty} [h_d(n)-h(n)]^2$$

or, since $h(n) = 0$, $n < 0$ and $n \geq N$,

$$\varepsilon^2 = \sum_{n=0}^{N-1} [h_d(n)-h(n)]^2 + \sum_{n=-\infty}^{-1} h_d^2(n) + \sum_{n=N}^{\infty} h_d^2(n) \ .$$

Clearly, the choice of $h(n)$ cannot affect the last summations. The first is non-negative and consequently its minimum value is zero which is achieved for $h(n) = h_d(n)$.

It should be stressed that although a rectangular window minimizes the mean square error, it does not generally result in the best frequency characteristic in terms, for example of passband or stopband ripple.

COMPUTATION OF THE DISCRETE FOURIER TRANSFORM - PART 1

1. Lecture 18 - 49 minutes

a.

Computation of the DFT

$$X(k)=\sum_{n=0}^{N-1} x(n) W_N^{nk}$$

$$x(n)=\frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-nk}$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

Direct Computation

$x(n) W_N^{kn} \Rightarrow$ 1 complex multiply (4 mults, 2 adds)

$X(k)$ $k=0,1,\ldots N-1$

N^2 complex multiplies

$N(N-1)$ complex adds

$\approx N^2$ MADS

Fast Fourier Transform (FFT) $N = P_1\ P_2 \cdots P_v$

Complex MADS $\propto N[P_1+P_2+\cdots P_v]$

$N = 2^v \Rightarrow N \log_2 N$

$$X(k)=\sum_{n=0}^{N-1} x(n) W_N^{nk}$$

$$=\underbrace{\sum x(n) W_n^{nk}}_{n \text{ even}} + \underbrace{\sum x(n) W_N^{nk}}_{n \text{ odd}}$$

$n=2r$ $\quad$ $n=2r+1$

$r=0,1,\ldots,\frac{N}{2}-1$

b.

$$X(k)=\sum_{r=0}^{\frac{N}{2}-1} x(2r) W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} x(2r+1) W_N^{(2r+1)k}$$

$$W_N^{(2r+1)k} = W_N^k W_N^{2rk}$$

$$W_N^2 = e^{-j\frac{2\pi}{N}\cdot 2} = e^{-j2\pi/(N/2)} = W_{N/2}$$

$$X(k)=\sum_{r=0}^{\frac{N}{2}-1} x(2r) W_{N/2}^{rk} + W_N^k \sum_{r=0}^{\frac{N}{2}-1} x(2r+1) W_{N/2}^{rk}$$

$$X(k) = \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x(2r) W_{N/2}^{rk}}_{\frac{N}{2}\text{ POINT DFT}} + W_N^k \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x(2r+1) W_{N/2}^{rk}}_{\frac{N}{2}\text{ POINT DFT}}$$

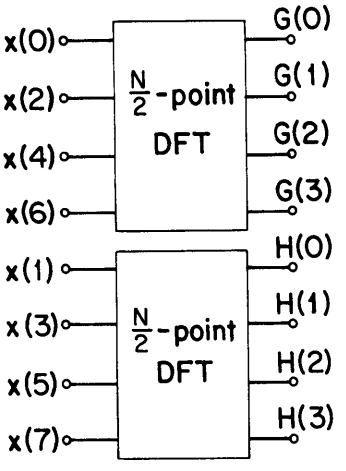
$G(k)$ $\qquad$ $H(k)$

DFT of a sequence in terms of the DFT of the even and odd numbered points.

$$X(k) = G(k) + W_N^k H(k)$$

$$2\left(\frac{N}{2}\right)^2 + N = N + \frac{N^2}{2}$$

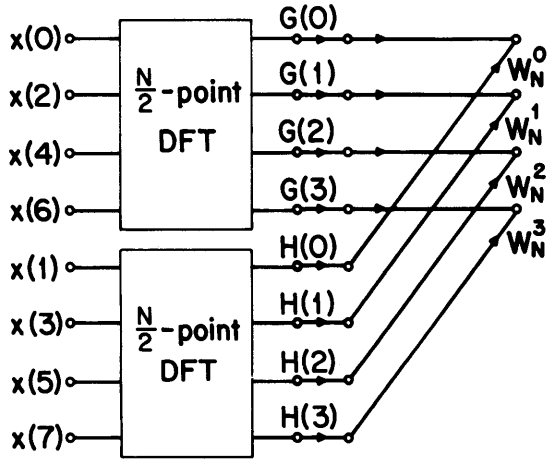
c.



$$X(k) = G(k) + W_N^k H(k)$$

N/2-point DFT's of even and odd-numbered points

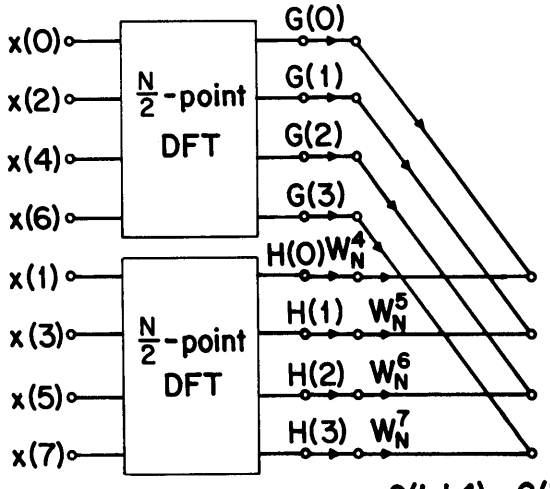
d.



$$X(k) = G(k) + W_N^k H(k)$$

Combination of G(k) and H(k) to obtain first half of X(k)

e.

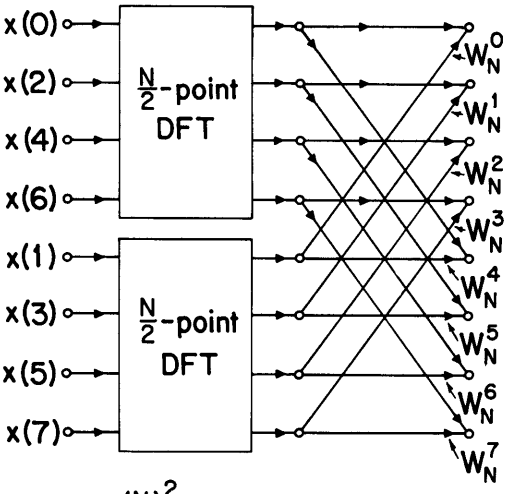


$$X(k) = G(k) + W_N^k H(k)$$

$$G(k+4) = G(k)$$
$$H(k+4) = H(k)$$

Combination of G(k) and H(k) to obtain second half of X(k)

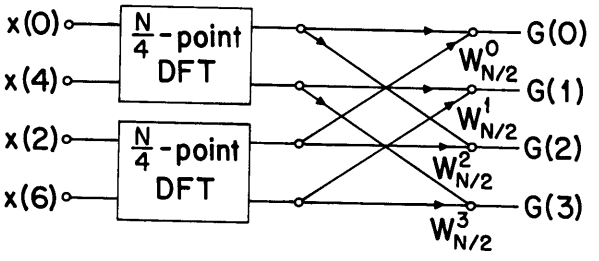
f.



$2\left(\frac{N}{2}\right)^2+N$ MADS

Combination of G(k) and H(k) to obtain X(k)

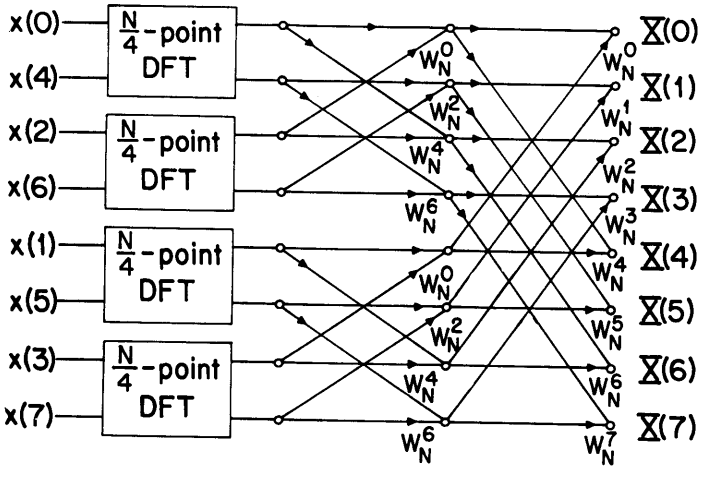
g.



$W_{\frac{N}{2}} = W_N^2$

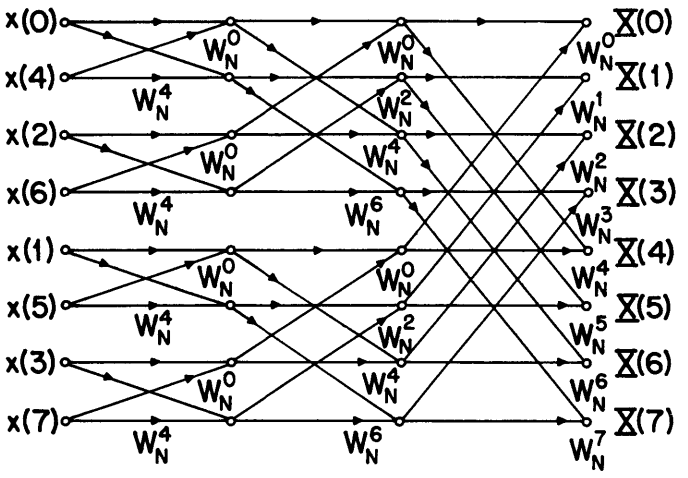
Computation of G(k) in terms of two N/4-point DFT's

h.



Computation of X(k) by combining flow-graph g and h.

i.



Flow-graph for computation of X(k)

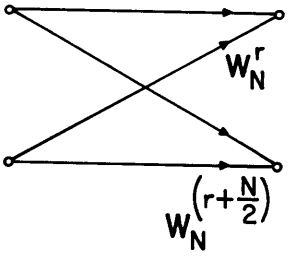
j.

$$N = 2^{\nu}$$

Savings in computation resulting from successive decimation in time

$$\begin{aligned}
&1 \text{ N POINT DFT} && N^2 \\
&2\ \tfrac{N}{2} \text{ POINT DFT'S} + && 2\left(\tfrac{N}{2}\right)^2 + N && \left(\tfrac{N}{2}\right)^2 \rightarrow 2\left(\tfrac{N}{4}\right)^2 + \tfrac{N}{2} \\
&4\ \tfrac{N}{4} \text{ POINT DFT'S} + && 4\left(\tfrac{N}{4}\right)^2 + N + N && \left(\tfrac{N}{4}\right)^2 \rightarrow 2\left(\tfrac{N}{8}\right)^2 + \tfrac{N}{4} \\
&8\ \tfrac{N}{8} \text{ POINT DFT'S} + && 8\left(\tfrac{N}{8}\right)^2 + N + N + N && \left(\tfrac{N}{8}\right)^2 \rightarrow 2\left(\tfrac{N}{16}\right)^2 + \tfrac{N}{8} \\
& && \vdots \\
& && \underbrace{N + N + \cdots + N}_{\nu \text{ TIMES}}
\end{aligned}$$

k.

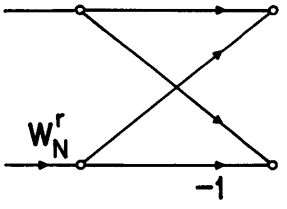


Basic butterfly computation in flow-graph j.

$$W_N^{\left(r+\frac{N}{2}\right)} = W_N^r \, W_N^{\frac{N}{2}}$$

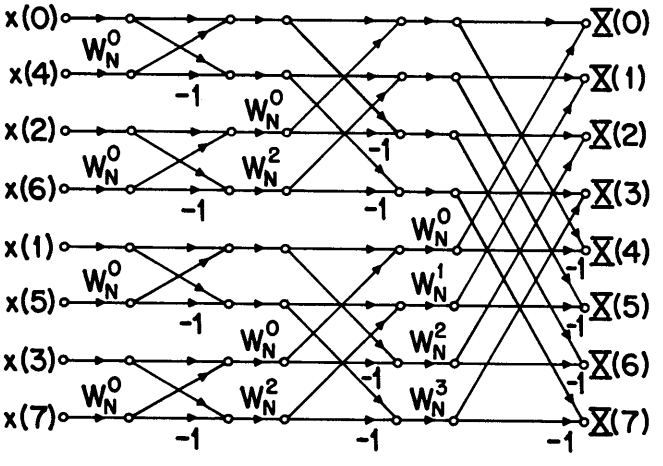
$$W_N^{\frac{N}{2}} = e^{-j\frac{2\pi}{N}\frac{N}{2}} = e^{-j\pi} = -1$$

l.



Rearrangement of the butterfly computation in l.

m.



Decimation-in-time FFT algorithm utilizing the butterfly computation in m.

n.

2. Comments

The most direct computation of the DFT requires an amount of computation propertional to N^2 where N is the length of the transform. However by exploiting symmetry and periodicity properties of the complex exponential sequences it is possible to reduce dramatically the amount of computation required. In this lecture we begin a discussion of the set of algorithms which result.

3. Reading

Text: Sections 9.0 (page 581), 9.1 and 9.3 up to (but not including) section 9.3.1.

4. Problems

Problem 18.1

In figure 9.3 of the text (page 589) is shown the flowgraph for implementing an eight-point DFT by first computing two four-point DFT's. Draw the corresponding flow-graph for implementing a sixteen-point DFT by first computing two eight-point DFT's.

Problem 18.2

Suppose that a program is available for evaluating a DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)kn}, \qquad k = 0,1,\ldots,N-1$$

Show how this same program can be used to compute the inverse DFT,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j(2\pi/N)kn}, \qquad n = 0,1,\ldots,N-1$$

Problem 18.3

Consider the FFT algorithm for N a power of 2, implemented in the form characterized by Figure 9.10 in the text. We are assuming that N is an arbitrary power of 2, not that N = 8. In indexing through the data in an array, we shall assume that points in an array are stored in consecutive complex (double) registers numbered 0 through N - 1. The arrays are numbered 0 through $\log_2 N$. The array holding the initial data is the zero$^{\underline{th}}$ array. The output of the first stage of butterflies is the first array, etc. All of the following questions relate to the computation of the m$^{\underline{th}}$ array where $1 \leq m \leq \log_2 N$. The answer should be in terms of m. All of the questions can be answered by generalizing the results for N = 8.

(a) How many butterflies are to be computed?

(b) What are the powers of W_N involved in computing the m$^{\underline{th}}$ array from the (m - 1)st array?

(c) What is the separation between the addresses of the two complex input points to a butterfly?

(d) What is the separation between the addresses of the first points of butterflies utilizing the same coefficients?

Problem 18.4

In computing the DFT of real sequences it is possible to reduce the amount of computation by utilizing the fact that the sequence is real. In particular, consider two real-valued sequences $x_1(n)$ and $x_2(n)$ with DFT's $X_1(k)$ and $X_2(k)$ respectively. Let $g(n)$ be the complex sequence given by $g(n) = x_1(n) + j\, x_2(n)$, and let $G(k)$ be its DFT. Let $G_{OR}(k)$, $G_{ER}(k)$, $G_{OI}(k)$ and $G_{EI}(k)$ denote, respectively, the odd part of the real part, the even part of the real part, the odd part of the imaginary part, and the even part of the imaginary part of $G(k)$. Using the fact that for a real sequence, the real part of its DFT is even and its imaginary part is odd (see Table 8.2, page 547 of the text), determine $x_1(k)$ and $x_2(k)$ in terms of $G_{OR}(k)$, $G_{ER}(k)$, $G_{OI}(k)$ and $G_{EI}(k)$.

The result derived in this problem is commonly utilized in one of two ways. If there are two real sequences for which we want the DFT, we can compute their transforms simultaneously and then separate the transforms using the result in the problem. As a second possibility, we can separate a real sequence into two subsequences, consisting of its even and odd numbered points, simultaneously compute the DFT of each of these subsequences, and then combine these to obtain the DFT of the total sequence according to equation (9.14) in the text.

COMPUTATION OF THE DISCRETE FOURIER TRANSFORM - PART 1

Solutions 18.1

The flow-graph of Fig. 9.3 of the text is based on the decomposition of X(k) in the form of equation 9.14 of text. For N = 16, the corresponding flow-graph expressing X(k) as a combination of two eight-point DFT's is shown in Figure S18.1-1 below.

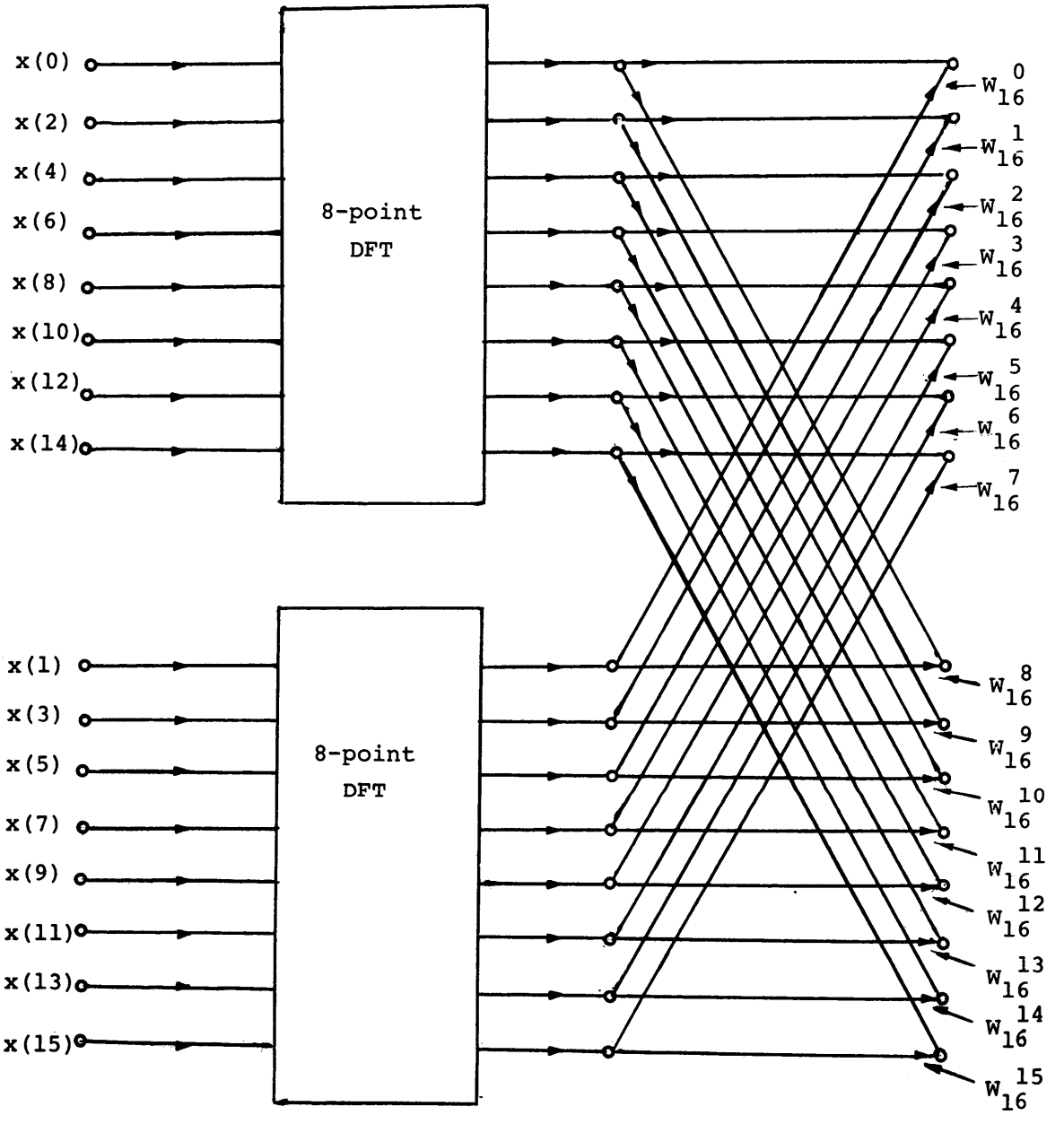


Figure S18.1-1

Solution 18.2

From the expression for the inverse DFT it follows that

$$N\, x^*(n) = \sum_{k=0}^{N-1} X^*(k)\, e^{-j(\frac{2\pi}{N})kn}$$

Thus by using as the input to the DFT program the complex conjugate of X(k), the output sequence will be N times the complex conjugate of x(n).

Solution 18.3

(a) $\frac{N}{2}$

(b) In proceeding from array (m - 1) to array m we are combining $2^{(m-1)}$ point DFT's to form 2^m point DFT's. Thus the coefficient are successive powers of W_M where $M = 2^m$. Thus these coefficients are W_M^k where $k = 0,1,2,\ldots(\frac{M}{2} - 1)$ or, since

$$W_M = (W_N)^{N/M}$$

The powers of W_N involved in computing the $m^{\underline{th}}$ array from the (m - 1)st array are

$$W_N^{Nk/M} \qquad k = 0,1,2,\ldots(M/2 - 1)$$

(c) $2^{(m-1)}$

(d) 2^m for $1 \leq m \leq (\log_2 N)-1$. For the last array ($m=\log_2 N$) there are no two butterflies utilizing the same coefficients.

Solution 18.4

With $g(n) = x_1(n) + j\, x_2(n)$,

$$G(k) = X_1(k) + j\, X_2(k)$$

With $X_1(k)$ and $X_2(k)$ expressed in terms of their real and imaginary parts,

$$X_1(k) = X_{1R}(k) + j\, X_{1I}(k)$$

$$X_2(k) = X_{2R}(k) + j\, X_{2I}(k)$$

G(k) can be written as

$$G(k) = [X_{1R}(k) - X_{2I}(k)] + j\, [X_{2R}(k) + X_{1I}(k)]$$

Now, since $x_1(n)$ and $x_2(n)$ are real, $X_{1R}(k)$ and $X_{2R}(k)$ are even and $X_{1I}(k)$ and $X_{2I}(k)$ are odd. Thus,

$$G_{ER}(k) = X_{1R}(k)$$

$$G_{OR}(k) = -X_{2I}(k)$$

$$G_{EI}(k) = X_{2R}(k)$$

$$G_{OI}(k) = X_{1I}(k)$$

COMPUTATION OF DISCRETE FOURIER TRANSFORM - PART 2

1. Lecture 19 - 49 minutes

a.

Decimation in Frequency

$$X(k)=\sum_{n=0}^{N-1}x(n)W_N^{nk}$$

$$=\sum_{n=0}^{\frac{N}{2}-1}x(n)W_N^{nk}+\sum_{n=\frac{N}{2}}^{N-1}x(n)W_N^{nk}$$

$$W_N^{\frac{N}{2}k}\sum_{n=0}^{\frac{N}{2}-1}x(n+\tfrac{N}{2})W_N^{nk}$$

$$X(k)=\sum_{n=0}^{\frac{N}{2}-1}\left[x(n)+(-1)^k x(n+\tfrac{N}{2})\right]W_N^{nk}$$

k even - $X(2r)$

$$\sum_{n=0}^{\frac{N}{2}-1}\left[x(n)+x(n+\tfrac{N}{2})\right]W_N^{2rn}$$

$r=0,1,2\ldots(\frac{N}{2}-1)$

k odd - $X(2r+1)$

$r=0,1,\ldots(\frac{N}{2}-1)$

$$\sum_{n=0}^{\frac{N}{2}-1}\left[x(n)-x(n+\tfrac{N}{2})\right]W_N^{n}W_N^{2rn}$$

$W_N^{2rn}=W_{\frac{N}{2}}^{rn}$

b.

k even

$$X(2r)=\sum_{n=0}^{\frac{N}{2}-1}g(n)W_{N/2}^{rn}$$

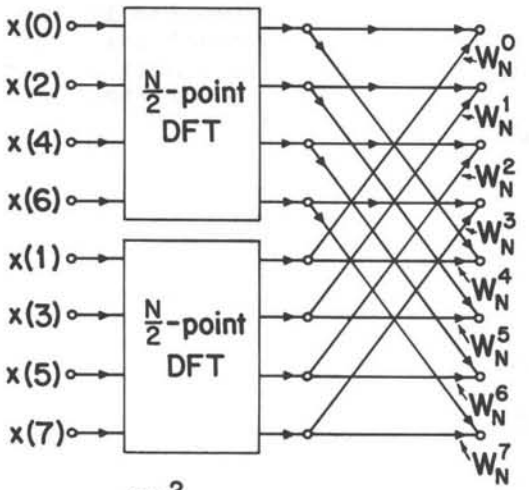
$$g(n)=x(n)+x(n+\tfrac{N}{2})$$

k odd

$$X(2r+1)=\sum_{n=0}^{\frac{N}{2}-1}\left[h(n)W_N^n\right]W_{N/2}^{rn}$$

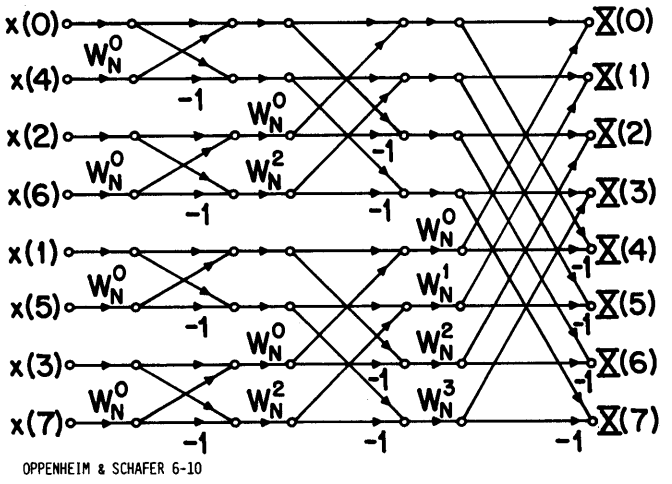
$$h(n)=x(n)-x(n+N/2)$$

c.



$2\left(\frac{N}{2}\right)^2+N$ MADS

Decomposition of an N-point DFT into 2 N/2-point DFT's.



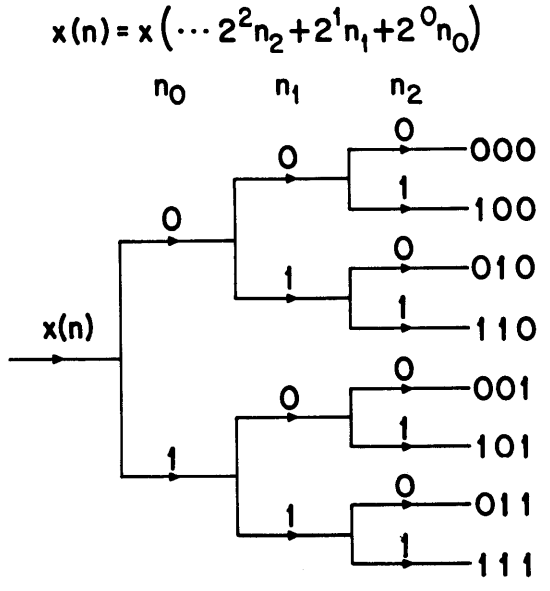
FFT flow-graph for decimation-in-time algorithm.

d.

	Storage Register		Data Index
0	000	X(0)	000
1	001	X(4)	100
2	010	X(2)	010
3	011	X(6)	110
4	100	X(1)	001
5	101	X(5)	101
6	110	X(3)	011
7	111	X(7)	111

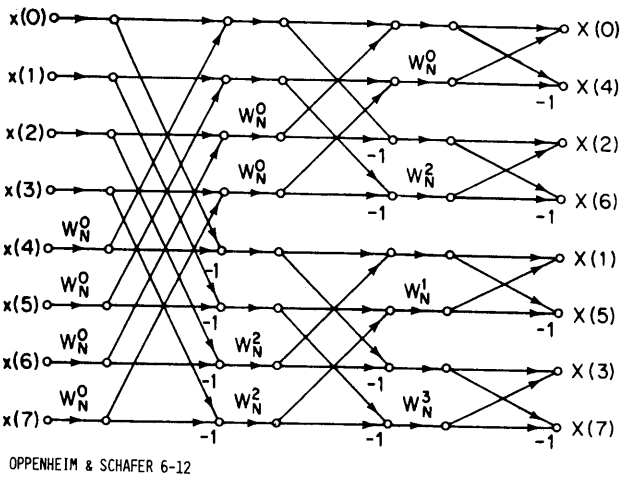
Relation between data index and data storage register.

e.



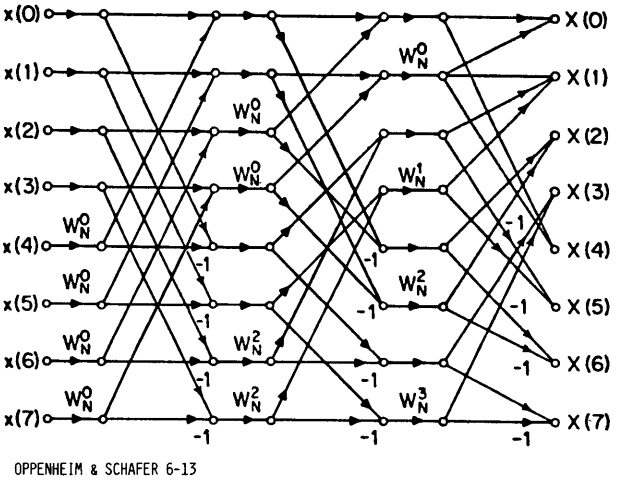
Sorting of data implied by the development of the decimation-in-time algorithm.

f.



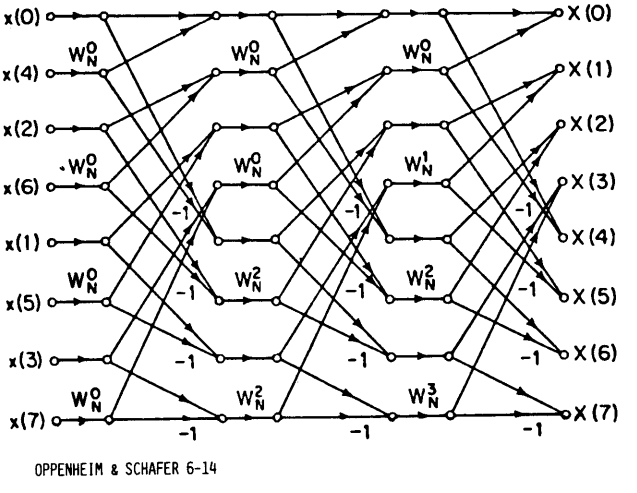
g.

Rearrangement of flow-graph d. with data in normal order and output in bit-reversed order.

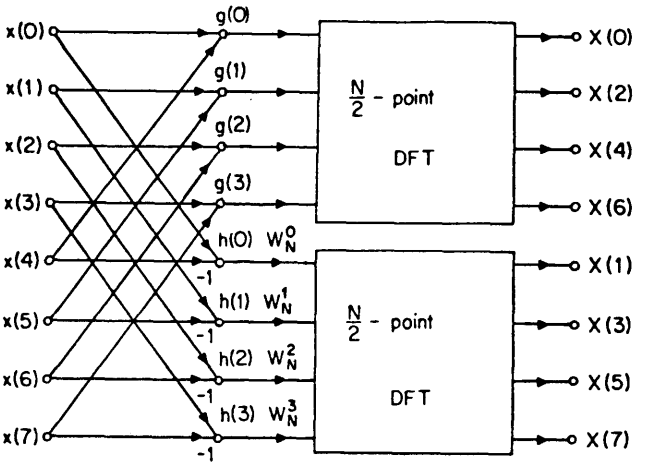


h.

Rearrangement of flow-graph d having the same geometry for each stage.

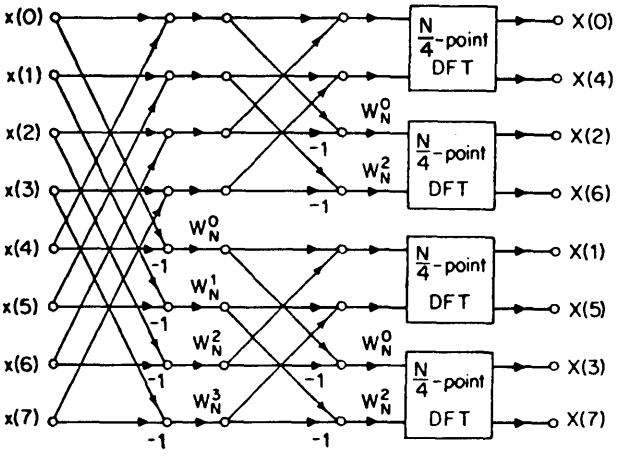


i.



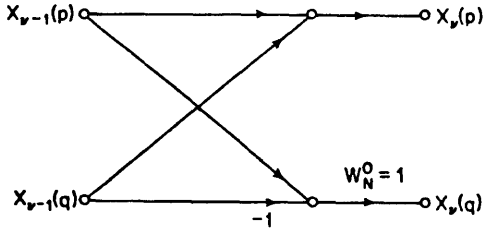
Computation of even and odd-numbered DFT values.

j.



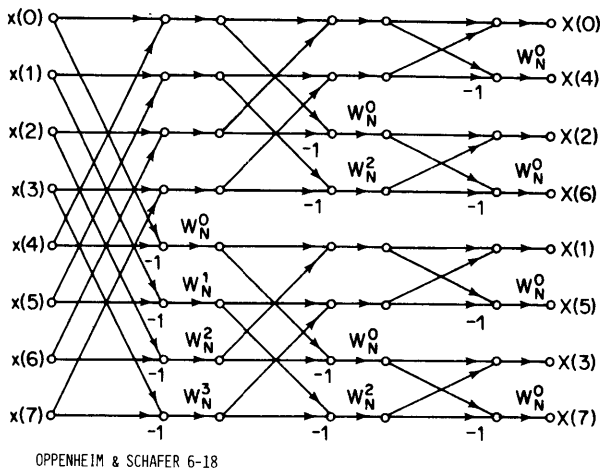
Decomposition of the N/2-point DFT's of flow-graph j.

k.



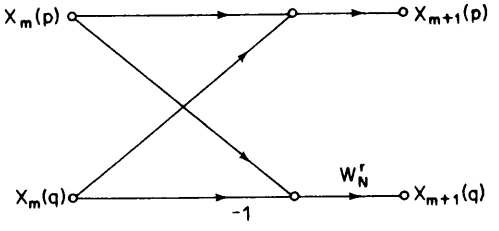
Flow-graph for two-point DFT.

l.



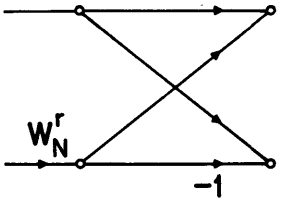
Flow-graph of complete decimation-in-frequency decomposition of an eight-point DFT computation.

m.



Flow-graph of a typical butterfly computation required in decimation-in-frequency FFT algorithm.

n.



Flow-graph of a typical butterfly computation required in decimation-in-time FFT algorithm.

o.

2. Comments

In this lecture we continue the discussion which was begun last time, in which we had developed a flow chart for efficient computation of the DFT. Here we discuss the implication of that flow chart for "in-place" computation if the input data is stored in a "bit reversed" order. This bit-reversed order arises as a natural consequence of the way in which that flow-graph was derived, i.e. by dividing the input into its even and odd numbered points, then dividing each of those in a similar manner, etc.

By rearranging the flow-graph it is possible to generate a number of other algorithms. For example, the input ordering can be rearranged so that bit reversal is not required. However, the advantage of in-place computation is lost. Another rearrangement, commonly referred to as the Singleton algorithm results in a flow-graph structure which is identical from stage to stage. This form is useful when the data is stored in sequential memory such as shift registers or disk rather than random access memory.

The above algorithms are collectively referred to as "decimation-in-time" algorithms because they were based on successive subdivisions of the input sequence. A companion set of algorithms referred to as "decimation-in-frequency" are based on successive subdivisions of the output. We conclude this lecture by introducing this class of algorithms, obtaining in particular, the basic flow-graph representing this algorithm. In the next lecture we consider a number of rearrangements of this flow-graph.

3. Reading

Text: Section 9.3.1 (page 592) and 9.3.2. Also section 9.4 up to, but not including section 9.4.1.

4. Problems

Problem 19.1

Consider a 16-point sequence x(0), x(1),..., x(15). List the sequence in bit-reversed order.

Problem 19.2

(a) Draw the flow-graph for a four-point decimation-in-time FFT algorithm utilizing the butterflies of Figure 9.9 of the text and with the input in bit-reversed order, the output in normal order, and

representing in-place computation.

(b) Rearrange the flow graph of part (a) so that it still corresponds to in-place computation but with the input in normal-order and the output in bit-reversed order.

Problem 19.3

Consider the FFT algorithm for N a power of 2, implemented in the form characterized by Figure 9.20 (page 602) in the text. We are assuming that N is an arbitrary power of 2, not that N=8. In indexing through the data in an array, we shall assume that points in an array are stored in consecutive complex (double) registers numbered 0 through N - 1. The arrays are numbered 0 through $\log_2 N$. The array holding the initial data is the zeroth array. The output of the first stage of butterflies is the first array, etc. All of the following questions relate to the computation of the m^{th} array where $1 \leq m \leq \log_2 N$. The answer should be in terms of m. All of the questions can be answered by generalizing the results for N = 8.

(a) How many butterflies are to be computed?

(b) What are the powers of W_N involved in computing the m^{th} array from the (m - 1)st array?

(c) What is the separation between the addresses of the two complex input points to a butterfly?

(d) What is the separation between the addresses of the first points of butterflies utilizing the same coefficients? Note that the butterfly computation for this algorithm is of the form of Fig. 9.21 in the text, i.e. the coefficient multiplication is applied at the output of the butterfly.

Problem 19.4*

When implementing a decimation-in-time FFT algorithm, the basic butterfly computation is as shown in the flow graph of Figure P19.4-1

$$X_{m+1}(p) = X_m(p) + W_N^r X_m(q)$$

$$X_{m+1}(q) = X_m(p) - W_N^r X_m(q)$$

In using fixed-point arithmetic in implementing the computations it is commonly assumed that all numbers are scaled to be less than unity. Thus we must be concerned with overflow in the butterfly computations.

(a) Show that if we require

$|X_m(p)| < \frac{1}{2}$ and $|X_m(q)| < \frac{1}{2}$

then overflow cannot occur in the butterfly computation; i.e.,

$\text{Re}\,[X_{m+1}(p)] < 1$, $\text{Im}\,[X_{m+1}(p)] < 1$, $\text{Re}\,[X_{m+1}(q)] < 1$,

and $\text{Im}\,[X_{m+1}(q)] < 1$

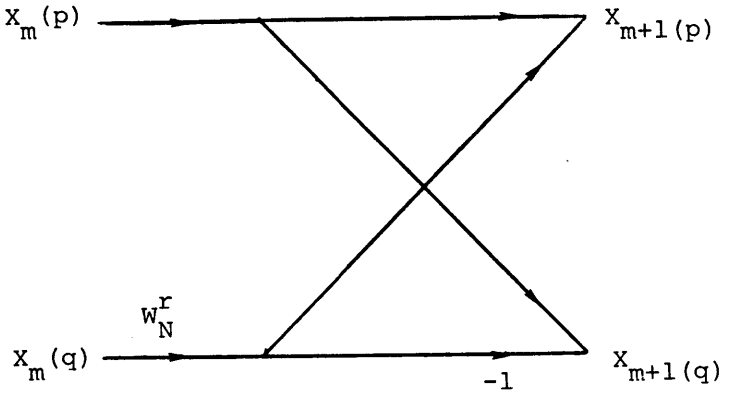


Figure P19.4-1

(b) In practice, it is easier and more convenient to require

$$|\text{Re}[X_m(p)]| < \frac{1}{2}, \quad |\text{Im}[X_m(p)]| < \frac{1}{2}$$

$$|\text{Re}[X_m(q)]| < \frac{1}{2}, \quad |\text{Im}[X_m(q)]| < \frac{1}{2}$$

Are these conditions sufficient to ensure that overflow cannot occur in the butterfly computation? Justify your answer.

Problem 19.5*

The FORTRAN program shown below implements the decimation-in-time algorithm of Figure 9.10 of the text. In the subroutine FFT(X, M), X is a complex array of dimension N that contains initially the input sequence x(n) and finally contains the transform X(k). The quantity M is an integer, $M = \log_2 N$.

(a) From a cursory inspection of the program indicate which lines of code are concerned with (1) bit reversal, (2) recursive computation of the complex exponential multipliers, and (3) the basic butterfly computation.

(b) Three errors have been inserted into the program as it is given here. Find these errors and make appropriate corrections to the FORTRAN code.

```
      SUBROUTINE FFT(X,M)                                              0001
      COMPLEX X(1024), U,W,T                                           0002
      N = 2**M                                                         0003
      NV2 = N/2                                                        0004
      NM1 = N - 1                                                      0005
      J = 1                                                            0006
      DO 7 I = 1,NM1                                                   0007
      T = X(J)                                                         0008
      X(J) = X(I)                                                      0009
      X(I) = T                                                         0010
5     K = NV2                                                          0011
6     IF(K.GE.J) GO TO 7                                               0012
      J = J - K                                                        0013
      K = K/2                                                          0014
      GO TO 6                                                          0015
7     J = J + K                                                        0016
      PI = 3.14159265358979                                            0017
      DO 20 L = 1,M                                                    0018
      LE = 2**L                                                        0019
      LE1 = LE/2                                                       0020
      U = (1.0,0.0)                                                    0021
      W = CMPLX(COS(PI/FLOAT(LE1)), SIN(PI/FLOAT(LE1)))                0022
      DO 20 J = 1, LE1                                                 0023
      DO 10 I = J,N,LE                                                 0024
      IP = I + LE                                                      0025
      T = X(IP)*U                                                      0026
      X(IP) = X(I) - T                                                 0027
10    X(I) = X(I) + T                                                  0028
20    U = U*W                                                          0029
      RETURN                                                           0030
      END                                                              0031
```

COMPUTATION OF THE DISCRETE FOURIER TRANSFORM - PART 2

Solution 19.1

NORMAL ORDER	BIT REVERSED ORDER
x(0000) = x(0)	x(0000) = x(0)
x(0001) = x(1)	x(1000) = x(8)
x(0010) = x(2)	x(0100) = x(4)
x(0011) = x(3)	x(1100) = x(12)
x(0100) = x(4)	x(0010) = x(2)
x(0101) = x(5)	x(1010) = x(10)
x(0110) = x(6)	x(0110) = x(6)
x(0111) = x(7)	x(1110) = x(14)
x(1000) = x(8)	x(0001) = x(1)
x(1001) = x(9)	x(1001) = x(9)
x(1010) = x(10)	x(0101) = x(5)
x(1011) = x(11)	x(1101) = x(13)
x(1100) = x(12)	x(0011) = x(3)
x(1101) = x(13)	x(1011) = x(11)
x(1110) = x(14)	x(0111) = x(7)
x(1111) = x(15)	x(1111) = x(15)

Solution 19.2

(a)

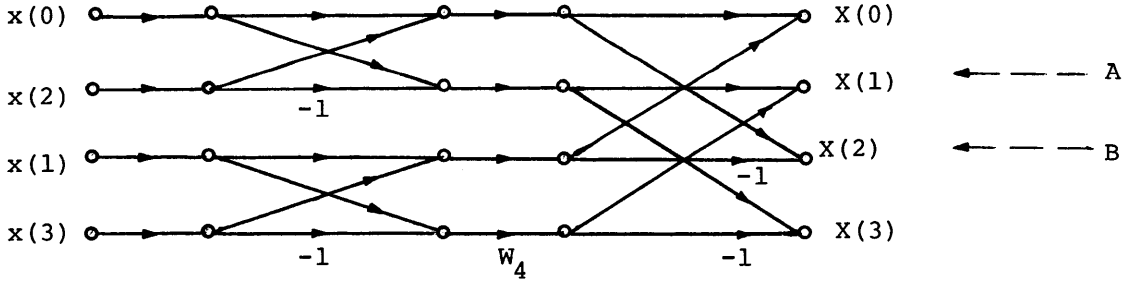


Figure S19.2-1

(b)

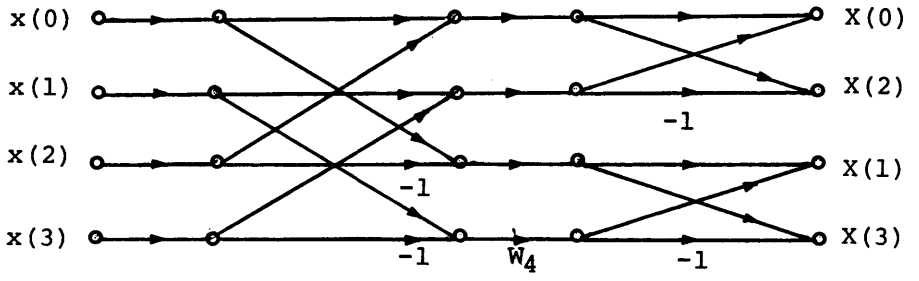


Figure S19.2-2

The desired flow-graph is obtained by interchanging lines A and B in Figure S19.2-1

Solution 19.3

(a) $N/2$

(b) $W_N^{Mk/2}$ $\quad k = 0,1,\ldots,N/M - 1 \quad M = 2^m$

(c) $N \cdot 2^{-m}$

(d) $N \cdot 2^{-m+1}$; $\quad 2 \leq m \leq \log_2 N$

Solution 19.4*

(a) $X_{m+1}(p) = X_m(p) + W_N^r X_m(q)$

$$|X_{m+1}(p)| \leq |X_m(p)| + |W_N^r X_m(q)| = |X_m(p)| + |X_m(q)|$$

Thus with $|X_m(p)|$ and $|X_m(q)|$ both less than 1/2,

$|X_{m+1}(p)| < 1$

and consequently $|X_{m+1}(p)|^2 < 1$

Finally, since $|X_{m+1}(p)|^2 = \left\{\text{Re } [X_{m+1}(p)]\right\}^2 + \left\{\text{Im } [X_{m+1}(p)]\right\}^2$

it follows that

$|\text{Re } [X_{m+1}(p)]| < 1$

$|\text{Im } [X_{m+1}(p)]| < 1$

In a similar manner it follows that

$|\text{Re}\ [X_{m+1}(q)]| < 1$

$|\text{Im}\ [X_{m+1}(q)]| < 1$

(b) $\text{Re}\ [X_{m+1}(p)] = \text{Re}\ [X_m(p)] + \text{Re}\ [W_N^r\ X_m(q)]$

Let $X_m(q)$ be expressed in polar form as $Ae^{j\theta}$

Then $\text{Re}\ [X_{m+1}(p)] = \text{Re}\ [X_m(p)] + A\cos(\theta - \frac{2\pi r}{N})$

$\text{Re}\ [X_m(p)]$ is constrained to be less than 1/2 and since the magnitudes of the real and imaginary parts of $X_m(q)$ are constrained to be less than 1/2, A must be less than $1/\sqrt{2}$. If $\theta = \frac{2\pi r}{N}$, then

$|\text{Re}\ [X_{m+1}(p)]| = |\text{Re}\ [X_m(p)] + A| < \frac{1}{2} + \frac{1}{\sqrt{2}}$

Since this is greater than unity, we see that the stated constraints will not guarantee that no over-flow occurs.

Solution 19.5*

(a)

(1) Bit reversal - lines 7 through 16

(2) Recursive computation of W_N's - line 29

(3) Basic Butterfly computation - lines 26 through 28

(b)

(1) Insert between lines 7 and 8: If (I.GE.J) GO TO 5

(2) Line 22 should read:

```
W = CMPLX(COS(PI/FLOAT(LEI)), - SIN(PI/FLOAT(LEl)))
```

(3) Line 25: IP = I + LEl

COMPUTATION OF THE DISCRETE FOURIER TRANSFORM - PART 3

1. Lecture 20 - 45 minutes

a.

Computational Considerations

① Inverse DFT

$$x(n)=\frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-nk}$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

DFT

$$X(k)=\sum_{n=0}^{N-1} x(n)W_N^{nk}$$

② Bit Reversal

0 1 2 3 4 5 6 7

0 4 2 6 1 5 3 7

b.

$N=P_1\cdot P_2\cdots P_v$ (q_1)

$N=P_1\cdot q_1$

Subsequence: every P_1^{th} point

P_1 sequences of length q_1

A B C A B C A B C A B C

$$X(k)=\sum_{n=0}^{N-1} x(n)W_N^{kn}$$

$$=\sum_{r=0}^{q_1-1} x(P_1r)W_N^{P_1rk}$$

$$+\sum_{r=0}^{q_1-1} x(P_1r+1)W_N^{(P_1r+1)k}$$

$+\cdots$

$$X(k)=\sum_{l=0}^{P_1-1} W_N^{lk}\sum_{r=0}^{q_1-1} x(P_1r+l)W_N^{P_1rk}$$

$$W_N^{P_1rk}=e^{-j\frac{2\pi}{\frac{N}{P_1q_1}}P_1rk}$$

$$=e^{-j\frac{2\pi}{q_1}rk}=W_{q_1}^{rk}$$

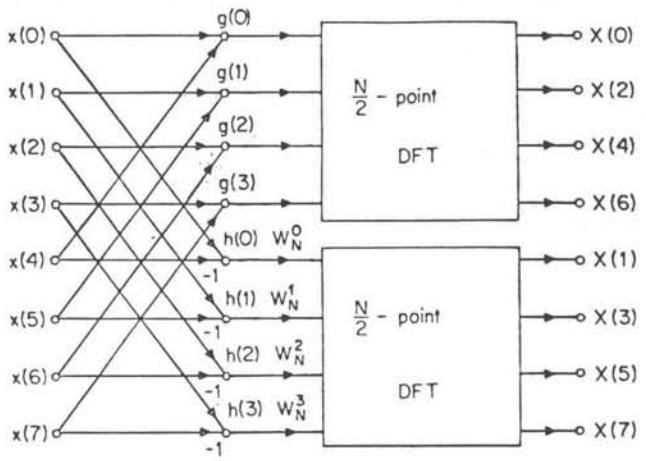
$$X(k)=\sum_{l=0}^{P_1-1} W_N^{lk}\underbrace{\sum_{r=0}^{q_1-1} x(P_1r+l)W_{q_1}^{rk}}_{q_1\text{-point DFT}}$$

$q_1=P_2\cdot\underbrace{P_3\cdots P_v}_{q_2}$

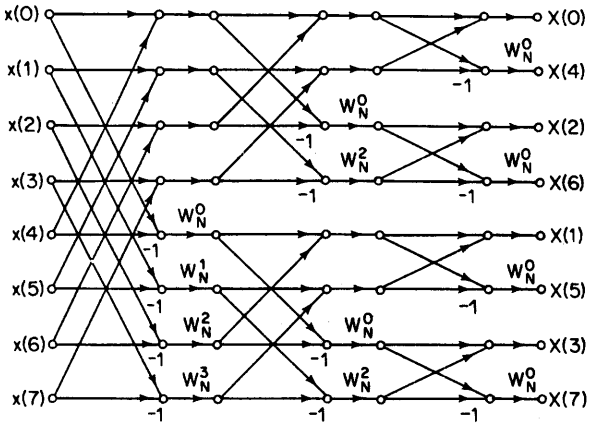
MADS $\Rightarrow$

$N[P_1+P_2+\cdots+P_v-v]$

c.

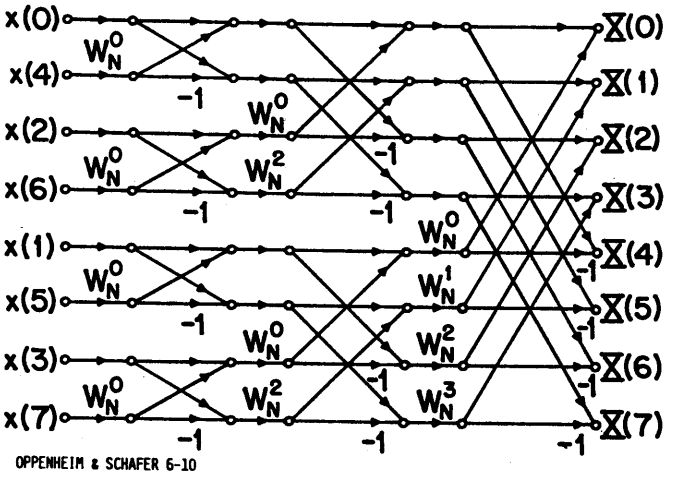


Computation of even and odd numbered DFT values.



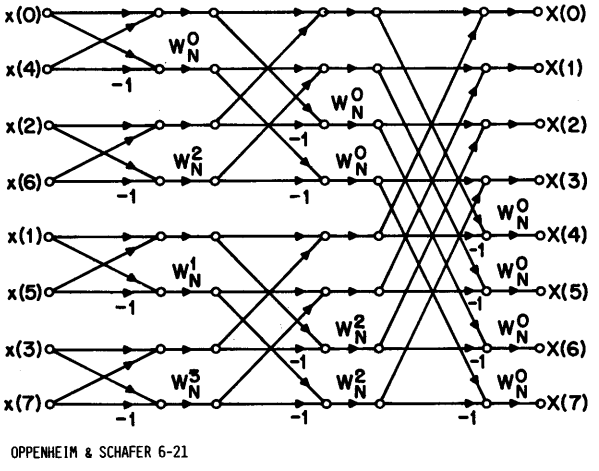
Flow-graph of complete decimation-in-frequency decomposition of an eight point DFT computation.

d.



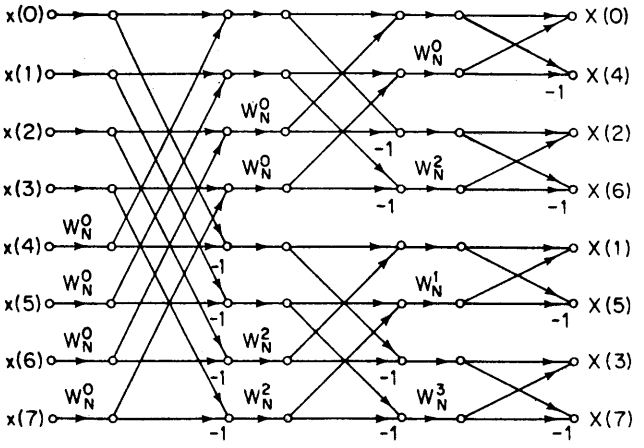
FFT flow-graph for decimation-in-time algorithm.

e.



Rearrangement of the decimation-in-frequency flow-graph d. The input is now in bit-reversed order and the output is in normal order.

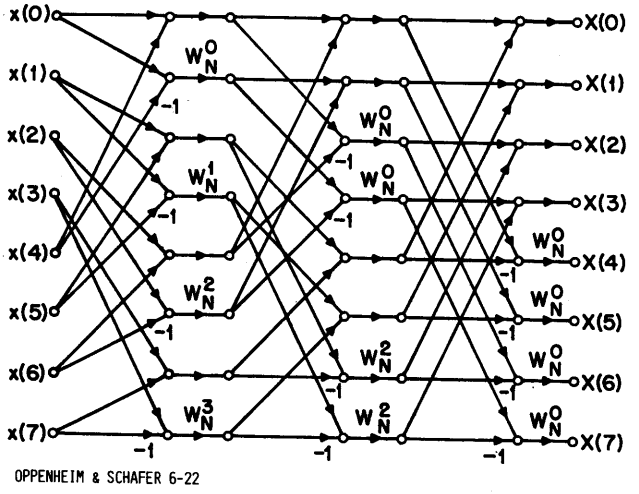
f.



OPPENHEIM & SCHAFER 6-12

Rearrangement of e so that the input is in normal order and output in bit-reversed order.

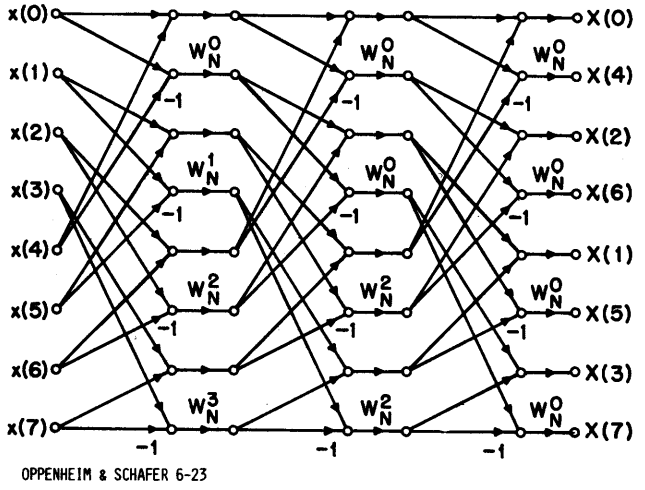
g.



OPPENHEIM & SCHAFER 6-22

Rearrangement of d so that both input and output are in normal order.

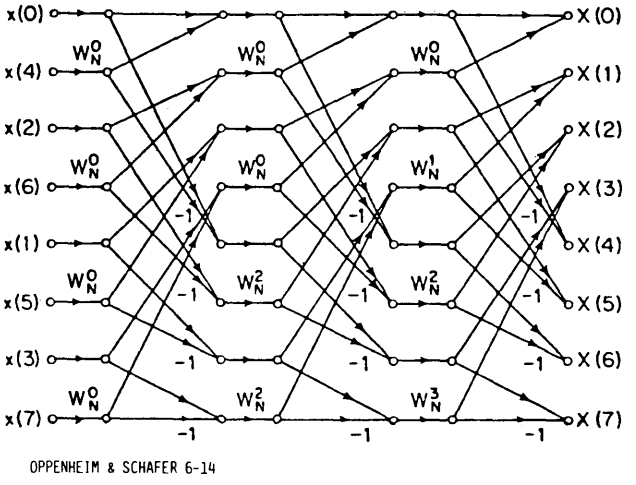
h.



OPPENHEIM & SCHAFER 6-23

Rearrangement of d so that geometry is identical in each stage.

i.



Decimation-in-time flow-graph for which the geometry is identical for each stage.

j.

Correction: On (Fig. a.) W_N is written as $e^{j\frac{2\pi}{N}}$ it should be $e^{-j\frac{2\pi}{N}}$.

2. Comments

In the previous lecture we introduced a class of FFT algorithms referred to as "decimation-in-frequency." This class of algorithms is developed on the basis of successive subdivisions of the output, as compared with the "decimation-in-time" algorithms which are based on successive subdivisions of the input. While the flowgraph for decimation-in-frequency as initially derived represents an in-place computation with the input in normal order and the output in bit-reversed order, there are a variety of rearrangements of the flowgraph just as there were for the "decimation-in-time" algorithms. The various forms of the "decimation-in-frequency" flowgraphs are related to the decimation-in-time" flowgraph through the transposition theorem. The choice between the various forms of the FFT algorithm is generally based on such considerations as the importance of in-place computation, whether it is more convenient to require bit-reversal at the input or output, order in which coefficients are stored, etc. For example if we intend to follow a DFT by an inverse DFT it is generally preferable to begin with an algorithm for which the input is in normal order and the output is in bit-reversed order. If for the inverse transform a form of the algorithm is used which requires bit-reversed input data and generates the output in normal order, it is never necessary to rearrange the order of the data.

Throughout the discussion of the FFT algorithms we have concentrated on "radix-2" algorithms, i.e. we have assumed that N is a power of two. More generally, efficient algorithms for the computation of the DFT can be utilized when N is decomposable as a product of factors. We

conclude this lecture with a brief introduction to these more general classes of FFT algorithms.

3. Reading

Text: Section 9.4 (page 599), and section 9.5.

4. Problems

Problem 20.1

The basic radix-2 FFT algorithms based on decimation-in-time are indicated in the text, Figures 9.10, 9.14, 9.15, 9.16. The basic radix-2 FFT algorithms based on decimation-in-frequency are indicated in the text Figures 9.20, 9.22, 9.23, 9.24. For each of these eight flow-graphs indicate whether or not each of the following following properties is true or not:

(1) Represents an in-place computation
(2) Input is in normal order
(3) Output is in normal order
(4) Coefficients should be stored in bit-reversed order
(5) Accessing of the data is identical for every array

Problem 20.2

We wish to implement a filter on a small computer by evaluating the DFT of input data, multiplying by the DFT of the unit-sample response and then computing the inverse DFT. The length of input data is a power of two but is sufficiently large that it cannot be stored in random-access memory. Consequently we wish to choose an algorithm which permits the data to be stored in and accessed from disk memory.

(a) How would you modify any one of the radix-2 algorithms discussed in the text so that it computes the inverse DFT rather than the DFT?

(b) Which of the eight radix-2 algorithms listed in problem 20.1 would it be most convenient to use for the computation of the DFT?

(c) Which of the eight radix-2 algorithms listed in problem 20.1 would you modify according to part (a) and use for the inverse DFT?

(d) In implementing the transform and inverse transform, let us assume that the disk is divided into four tracks which we'll refer to as A, B, C, and D. The input data will initially be stored on

tracks C and D. With the algorithm which you chose in (b), how should the input data be divided between tracks A and B?

(e) After completing the computation of the DFT, on what disk tracks and in what order is the result stored?

(f) Assume that the DFT values have been multiplied by the DFT of the unity sample response and the product is stored in the same order as were the DFT values in (c). Do these values have to be rearranged in any way before utilizing the algorithm chosen in (c) for the inverse DFT?

Problem 20.3

In rearranging data from normal order to bit-reversed order, a common procedure is to program a counter which counts sequentially in normal order and a second counter which counts sequentially in bit-reversed order. On most computers, of course, a normal counter usually corresponds to an index register which is incremented by unity. Most computers do not have a bit reversed counter but they are easy to implement. One possible flow-chart to implement a bit-reversed counter is show in Figure P20.3-1.

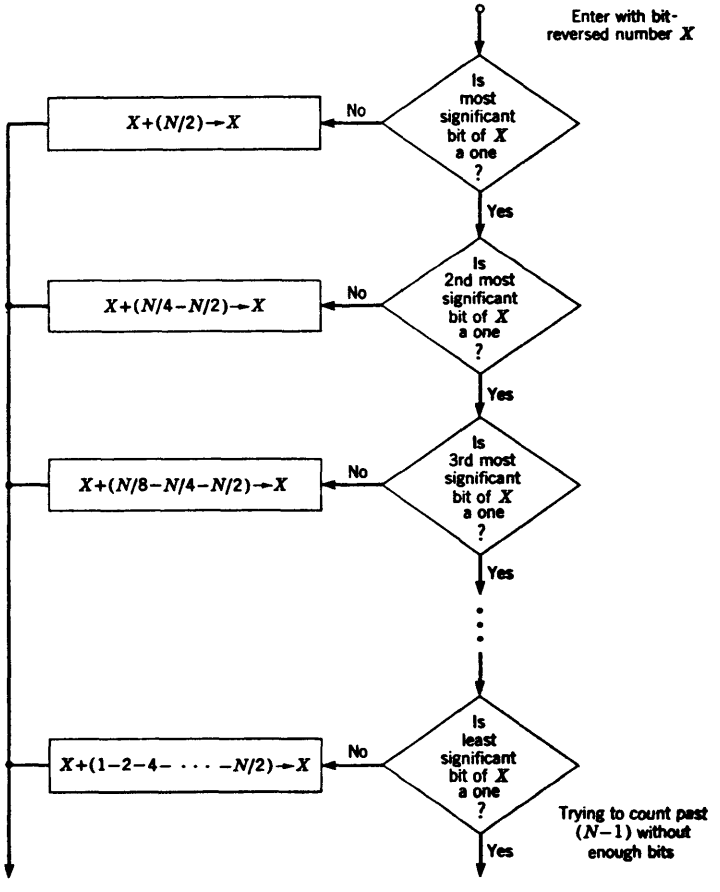


Figure P20.3-1

Let us assume that we have a normal counter (counter A) and a bit-reversed counter (counter B). In Figure P20.3-2 is shown a flow-chart intended to sort data from normal order to bit-reversed order. Determine whether a program implementing this flow-chart will sort the data as desired. If not, insert the necessary corrections into the flow-chart.

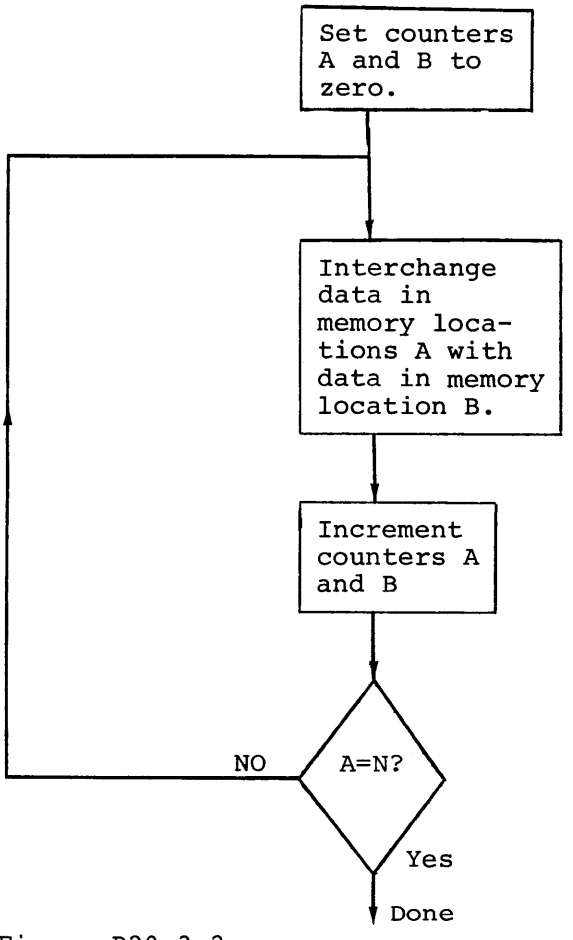

Set counters A and B to zero.
Interchange data in memory locations A with data in memory location B.
Increment counters A and B
A=N?
NO
Yes
Done

Figure P20.3-2

Problem 20.4

Draw the flow diagram for a nine-point (i.e., 3 x 3) decimation-in-time FFT algorithm.

Problem 20.5*

The FORTRAN program shown in Figure P20.5-1 is an implementation of the decimation-in-frequency algorithm depicted in Figure 6.18 of the text. The program evaluates the DFT:

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)kn}, \quad k = 0, 1, \ldots, N-1$$

```
      SUBROUTINE FFT(X,M)                                              0001
      COMPLEX X(1024), U,W,T                                           0002
      N = 2**M                                                         0003
      PI = 3.14159265358979                                            0004
      DO 20 L = 1,M                                                    0005
      LE = 2**(M + 1 - L)                                              0006
      LE1 = LE/2                                                       0007
      U = (1.0,0.0)                                                    0008
      W = CMPLX(COS(PI/FLOAT(LE1)), -SIN(PI/FLOAT(LE1)))               0009
      DO 20 J =1, LE1                                                  0010
      DO 10 I = J,N,LE                                                 0011
      IP = I+LE1                                                       0012
      T = X(I) + X(IP)                                                 0013
      X(IP) = (X(I) - X(IP))*U                                         0014
10    X(I) = T                                                         0015
20    U = U*W                                                          0016
      NV2 = N/2                                                        0017
      NM1 = N - 1                                                      0018
      J = 1                                                            0019
      DO 30 I = 1,NM1                                                  0020
      IF(I.GE.J) GO TO 25                                              0021
      T = X(J)                                                         0022
      X(J) = X(I)                                                      0023
      X(I) = T                                                         0024
25    K = NV2                                                          0025
26    IF(K.GE.J) GO TO 30                                              0026
      J = J - K                                                        0027
      K = K/2                                                          0028
      GO TO 26                                                         0029
30    J = J + K                                                        0030
      RETURN                                                           0031
      END                                                              0032
```

Figure P20.5-1

In the subroutine FFT(X,M) X is a complex array of dimension N that contains initially the input sequence x(n) and finally contains the transform X(k). The quantity M is an integer, $M = \log_2 N$.

This program is a straightforward implementation of the flow-graph of Fig. 9.20 of the text. The program is very elegant but not as efficient as it could be. Greater efficiency can be obtained at the cost of a more complex program.

A significant increase in efficiency is suggested by noting that in the last stage of the flow graph in Fig. 6.18, the complex multipliers are all unity. Thus if the last stage is implemented separately, we can eliminate N/2 complex multiplications

(a) What is the percentage reduction in multiplications that results?

(b) Modify the program to implement this saving in multiplications.

(c) Many small computers have FORTRAN compilers without the capability of complex arithmetic. Modify the given program so that only real operations are involved. That is, using the present subroutine as a guide, write a subroutine

FFT(XR,XI,M)

where XR and XI are real arrays of dimension N which initially contain the real part and the imaginary part of the input and finally the real and imaginary parts of the transform.

SOME CONCLUDING REMARKS

With lesson 20 we conclude this introductory set of lessons on digital signal processing. It has often been said that the purpose of a course is to uncover rather than to cover a subject and that description applies particularly in this case. Throughout the lectures I have tried to concentrate on the basic fundamentals of digital signal processing to provide a firm background for proceeding to applications and advanced topics. As you know, we have only covered the first six chapters in the text and omitted the more advanced topics from some of those. I would like to encourage you to take the time to look over the parts of those six chapters which were not assigned reading. I would also like to encourage you to continue on through the text; in chapters 10 and 11, in particular, you will find frequent reference to applications.

With the first six chapters as background, I feel that you will find much of the technical literature in the field of digital signal processing to be interesting and understandable. As I mentioned in the introductory lecture, this material has important applications in a wide variety of areas, and I would like to encourage you to explore some of these applications and also consider whether some of the techniques that we have discussed have application to your own area of interest. While we have only been able to present the fundamentals, I hope that this set of lectures will serve to open the door for you to an exciting, dynamic, and important field.

COMPUTATION OF THE DISCRETE FOURIER TRANSFORM - PART 3

Solution 20.1

	6.10	6.12	6.13	6.14	6.18	6.21	6.22	6.23
(1)	yes	yes	no	no	yes	yes	no	no
(2)	no	yes	yes	no	yes	no	yes	yes
(3)	yes	no	yes	yes	no	yes	yes	no
(4)	no	yes	no	no	no	yes	no	no
(5)	no	no	no	yes	no	no	no	yes

Solution 20.2

(a) As discussed in the lecture, the coefficients in the inverse DFT are the complex conjugates of the coefficients in the DFT. In addition the inverse DFT has associated with it a scale factor of 1/N. Consequently to modify any of the radix-2 algorithms to implement an inverse DFT, we replace the coefficients by their complex conjugates and apply a scale factor of (1/N) to the output. An alternative is to use the result of problem 2 lesson 18.

(b) and (c) The two algorithms which are particularly convenient when data is to be stored in and accessed from sequential memory such as disk are represented by the flow-graphs of Figures 9.16 and 9.24 of the text. Since the flow-graph of Figure 9.24 has data in normal order as its input, and the flow-graph of Figure 9.16 of the text has data in normal order as its output , it is most convenient to use the flow-graph of Figure 9.24 for the computation of the DFT and the flow-graph of Figure 9.16 for the computation of the inverse DFT.

(d) In the algorithm of Figure 9.24, successive points in an array are computed by combining data from the first half and last half of the previous array. Consequently the first half of the input data should be stored on track A and the second half on track B. In

computing the first array, the first half of the computed points are stored in track C and the second half on track D. In computing the second array we combine the data on tracks C and D, storing the first half of the results on track A and the second half on track B.

(e) There are $\log_2 N$ arrays to be computed. Thus, if $\log_2 N$ is even, the results are on tracks A and B. If $\log_2 N$ is odd, the results are on tracks C and D. Furthermore, the DFT values are stored on these tracks in bit-reversed order. For $\log_2 N$ even, the even numbered points are in bit-reversed order on track A and the odd numbered points are in bit-reversed order on track B.

(f) The algorithm of Figure 9.16 which is to be used for the inverse transform assumes that the input is in bit-reversed order. Consequently it is not necessary to sort the DFT values into normal order. In the algorithm of Figure 9.16 the computation of an array involves combining successive points in the previous array to obtain results in the first half and last half of the array being computed. Assume that the results from the computation of the DFT are stored on tracks A and B. As discussed in (e), the first half of these points (the even numbered points) are on track A and the second half (the odd numbered points) are on track B.

In implementing the flow-graph of Figure 9.16 we first combine successive points from track A generating results for the first and last half of the first array. When track A is completed, we combine successive points from track B. Thus in using the output of the flow-graph of Figure 9.24 as the input to the flow-graph of Figure 9.16 no rearrangement of data is necessary. The way in which data is accessed and stored is however different for the two algorithms.

Solution 20.3

Let m denote a memory location ($0 \leq m \leq N - 1$) and $\hat{m}$ the corresponding bit-reversed location. According to the flow-chart of Figure P20.3-2 when counter A is equal to m, the data in locations m and $\hat{m}$ are interchanged and when counter A is equal to $\hat{m}$ the data in locations m and $\hat{m}$ are again interchanged. Consequently at the end of the program in Figure P20.3-2 the data is still in normal order. To correct this it is necessary to ensure that an exchange between a memory location and its bit-reversed counterpart is made only once. One possible correction to the flow-chart of Figure S20.3-1 is indicated in Figure S20.3-1. This correction also takes into account the obvious fact that when $m = \hat{m}$ no exchange is necessary.

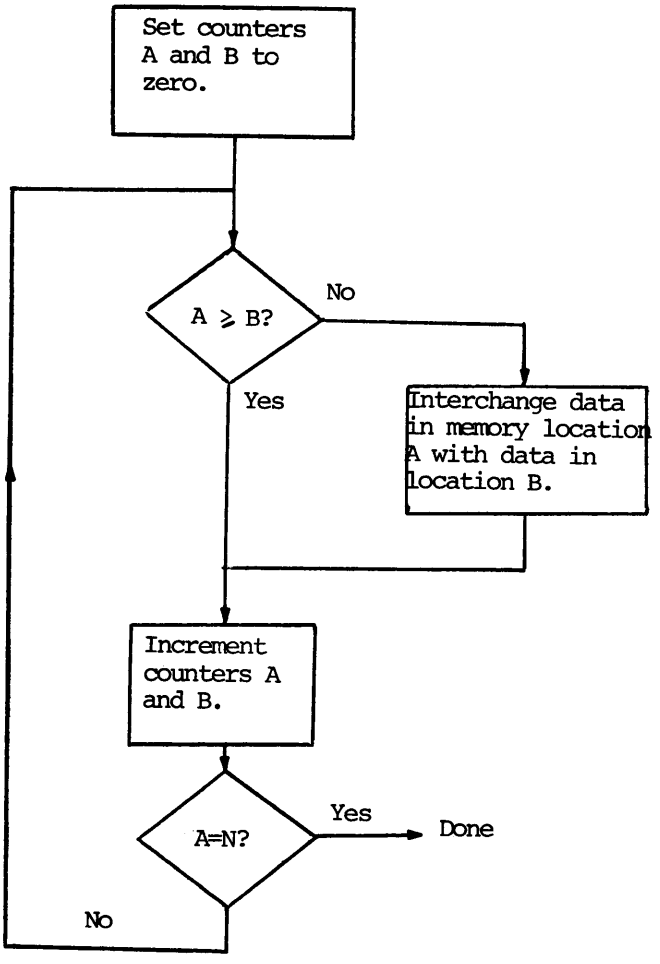


Figure S20.3-1

Solution 20.4

With N = 9 we divide x(n) into three subsequences each containing three points as indicated in Figure S20.4-1

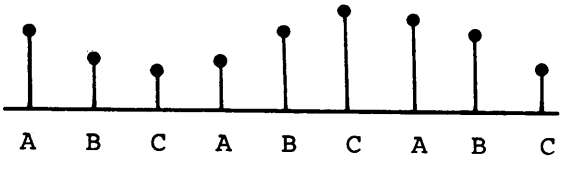


Figure S20.4-1

Thus,

$$X(k) = \sum_{n=0}^{8} x(n) W_9^{nk} = \underbrace{\sum_{r=0}^{2} x(3r) W_9^{3rk}}_{\text{Subsequence A}} + \underbrace{\sum_{r=0}^{2} x(3r+1) W_9^{(3r+1)k}}_{\text{Subsequence B}}$$

$$+ \underbrace{\sum_{r=0}^{2} x(3r+2) W^{(3r+2)k}}_{\text{Subsequence C}}$$

Using the fact that $W_9^3 = W_3$

$$X(k) = \sum_{r=0}^{2} x(3r)\, W_3^{rk} + W_9^k \sum_{r=0}^{2} x(3r+1)\, W_3^{rk} + W_9^{2k} \sum_{r=0}^{2} x(3r+2)\, W_3^{rk}$$

The resulting flowgraph is shown in Figure S20.4-2.

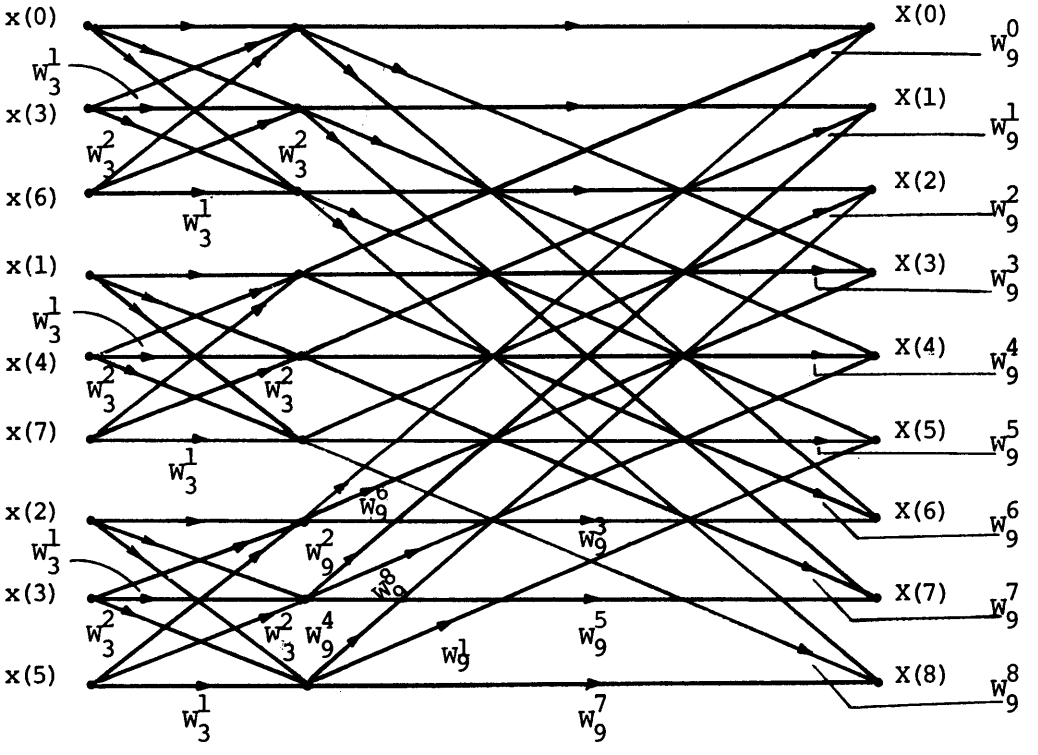


Figure S20.4-2

Solution 20.5*

(a) Assuming that we implement as multiplies all multiplications by W_N^o in the flow-graph of Figure 9.20 of the text there are a total of $\frac{N}{2} \log_2 N$ complex multiplications required. Thus by eliminating $\frac{N}{2}$ complex multiplications the percentage reduction is

$$\frac{(N/2)}{(\frac{N}{2})\log_2 N} \cdot 100\% = \frac{1}{\log_2 N} \cdot 100\%$$

(b) Change line 5 to DO 20 L = 1, M - 1

insert after line 16:

```
      DO 22 I=1,N,2
      IP=I+1
      T=X(I)+X(IP)
      X(IP)=X(I)-X(IP)
22    X(I)=T
```

(c)

```
      SUBROUTINE FFT(XI,XR,M)
      DIMENSION XI(1024), XR(1024)
      N=2**M
      PI=3.14159265358979
      DO 20 L=1,M
      LE=2**(M+1-L)
      LE1=LE/2
      UR=1.0
      UI=0.0
      WR=COS(PI/FLOAT(LE1))
      WI=-SIN(PI/FLOAT(LE1))
      DO 20 J=1,LE1
      DO 10 I = J,N,LE
      IP=I+LE1
      TR=XR(I)+XR(IP)
      TI=XI(I)+XI(IP)
      TMR=XR(I)-XI(IP)
      TMI=XI(I)-XI(IP)
      XR(IP)=TMR*UR-TMI*UI
      XI(IP)=TMR*UI+TMI*UR
      XR(I)=TR
10    XI(I)=TI
      TR=UR*WR-UI*WI
      UI=UR*WI+UI*WR
20    UR=TR
      NV2=N/2
      NM1=N-1
      J=1
      DO 30 I=1,NM1
      IF(I.GE.J) GO TO 25
      TR=XR(J)
      TI=XI(J)
```

```
        XR(J)=XR(I)
        XI(J)=XI(I)
        XR(I)=TR
        XI(I)=TI
25      K=NV2
        IF(K.GE.J) GO TO 30
        J=J-K
        K=K/2
        GO TO 26
30      J=J+K
        RETURN
        END
```